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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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Contributors to Part I will be presented 25 copies of reprints of their papers. Extra copies will be supplied, if desired, at net cost.

All contributions should be written legibly on one side only of the paper, and all diagrams should be neatly and accurately drawn on separate slips.

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ERRATA FOR THE PAPER

'ON NUMBERS WHICH CONTAIN NO FACTORS OF THE FORM $p(kp+1)$.'

By Sivasankaranarayana Pillai.

(Vol. XVIII, pp. 51—59).

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In the proof of Theorem I, I have assumed by mistake that

$$\pi_{i,1}\left(\frac{x}{p_i}\right) \sim \frac{1}{\phi(p_i)} \times \frac{x}{p_i \log (x/p_i)}$$

is true, even when p_i is a function of x, which is not admissible. Hence the following corrections should be made in the reading of Theorem I.

(1) In the enunciation, (page 51, line 9)

for $\quad$ '$N(x) > c \cdot \frac{x}{\log x} \log\log x$,'

read $\quad$ '$N(x) \neq O\left(\frac{x}{\log x}\right)$.

(2) *For* 'Hence if $p_r \leqslant \sqrt{x} < p_{r+1}$' (page 51, line 20).

read 'If r and δ (however small) are given, we can find $x_0 = x_0(r, \delta)$ so that

$$\pi_{i,1}\left(\frac{x}{p_i}\right) - \frac{1}{\phi(p_i)}\ \frac{x}{p_i \log (x/p_i)} = O\left(\frac{x\delta}{\phi(p_i)\, p_i \log (x/p_i)}\right)$$

for all $i \leqslant r$). Hence if r is any given positive integer, for all $x \geqslant x_0(r, \delta)$.'

(3) For the last three lines, (page 52, lines 5, 6, 7) in the proof *read* 'Now p_r increases beyond all limit with r, and r can be taken as great as we please. Hence

$$N(x) \neq O\left(\frac{x}{\log x}\right).'$$

I express my gratitude to my friend Mr. S. D. Chowla for pointing out the mistake.

To face Part II, page 71.

H. BALAKRAM, M.A., I.C.S.

1876—1929.

REMARKS ON MR. DURAIRAJAN'S PAPER ON PROPERTIES OF A NET OF PLANE CUBICS WITH GIVEN ASYMPTOTES

BY A. NARASINGA RAO.

Mr. Durairajan's investigation of the properties of the g-point of a line and of the g-lines through a point, is of special interest as furnishing a simple metrical example of a rather peculiar degeneration of the Geiser involution due to the coalescence of four of its fundamental elements.

Since on each line u there is, in general, a unique g-point, and through this point a defintė g-line v other than u, it follows that the two g-lines through any given point are corresponding elements in a Cremona line-involution. This involution is of *class* one, that is, there is but one pair of homologues through each point, and hence it must be a Geiser involution or one of its degenerations*.

The Geiser line-involution is defined by seven lines, $l_r (r = 1, 2...7)$. A pair of homologues u, v have the property that there exist an infinity of class cubics touching u, v, and the seven lines l_r. The involution is of order eight, with the lines l_r as fundamental elements of the third order, and carries any pencil of lines into tangents to a class 8-*ic* with the lines l_r for triple tangents.

The Geiser involution may be specialised by the coincidence of some of its fundamental lines. Thus consider all class cubics having a given line l for bitangent, and touching three other lines l_5, l_6 and l_7. These cubics will form a net, if the two contact points with l are not bound, but have one degree of freedom, the single condition upon them being linear in the co-efficients of the class cubic. Thus the two points of

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* Hudson, H. P.: *Cremona Transformations in plane and in space.* (1927). page 93.

The totality of lines in the plane each associated with its g-point form, in the terminology of Clebsch, a (1, 2) *Hauptcoincidenz*. For a brief sketch of the properties of such a coincidence, and in particular for a proof that the two lines through a point correspond in a Geiser involution, reference may be made to §§ 5, 6 of the paper which follows.

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contact may be required to be apolar to a definite pair of points on the line. Of the 9 common tangents of any two cubics of the net, four are absorbed by the bitangent l, and the other five consist of l_5, l_6, l_7 and two other lines u, v. These two lines (as also the other seven) are touched also by every member of the tangential pencil determined by the two cubics, and are homologues in a Geiser involution whose fundamental elements consist of l_5, l_6, l_7, and l repeated four times. It should be noted however that to specify the involution completely, we should know also the involution of the points of contact with l.

The involution of g-lines as defined in Mr. Durairajan's paper is precisely of this type, the line l being the line at infinity, and l_5, l_6 and l_7 the "three given asymptotes." The net of class cubics referred to above, is the net (11) of page 63, inscribed to the reference triangle ABC, and having the line at infinity for a bitangent, the points of contact being apolar to the Hessian pair of the points at infinity on the three sides.

It thus appears from Theorem 3 and Theorem 1, Cor. (ii), that the class sextic which is the envelope of self-corresponding lines in a general Geiser involutition, is here the square of a class cubic, and that the order quartic which is the locus of points the two homologues through which coincide, is here a repeated conic.

To obtain the transform of a point, we take the g-point of every line through it, and take the other g-line through each of these points. When the line is parallel to a side of the triangle ABC, the corresponding g-point is at infinity, and the other g-line through the point is indeterminate and is any one of a pencil of lines parallel to that side. Hence the three pencils of lines parallel to the three sides figure as a common factor in the transform of every point, and may be rejected the order of the transformation thus reducing by three.

The involution of g-lines through a point, is therefore of order not higher than five. That it is actually of order five may be shown by considering the transform of any vertex, say A. Of the lines through A, two are singular lines, namely, the sides AB and AC, the g-points of these lines being indeterminate. If M be a point on AB, the two g-lines through M are AB itself, and a line which meets CA and CB in points quidistant from M. The envelope of this latter line is a parabola P_3 touching AB at its middle-point and touching also the other two sides.

Thus the two lines AB, AC through A give rise to two parabolas P_3 and P_2. The g-points on the other lines through A, trace a line parallel to BC, and the other g-lines through these points are all parallel to the median bisecting BC. The transform of A is thus a composite curve of class five, consisting of the two parabolas and a point at infinity, and the homoloidal net of the transformation is that determined by the transforms of the three vertices. As these have no curve in common, the transformation is of order five.

It is clear from the above reasoning that the equations of transformation are of the form :

$$T: \qquad l' : m' : n' = k_1\, P_2P_3L_1 : k_2P_3P_1L_2 : k_3P_1P_2L_3$$

where

$$P_1 \equiv lm + ln - 2mn, \quad P_2 \equiv ml + mn - 2nl, \quad P_3 \equiv nl + nm - 2lm,$$

correspond to the three parabolas,

$$L_1 \equiv 2l - m - n, \quad L_2 \equiv 2m - l - n, \quad L_3 \equiv 2n - l - m,$$

correspond to the points at infinity on the three medians, and k_1, k_2, k_3 are certain constants, which may be determined by an examinations of the transform of a particular line. Thus the condition that the line $(1, 1, \lambda)$ transforms into $(1, 1, 1)$ whatever λ may be gives $k_1 = k_2 = k_3$, so that this k's may be omitted altogether from the equations T.

As L_r represents the point at infinity on the parabola P_r, it follows that the homoloidal net of T consists of curves touching the line at infinity in the same three points. The transformation T is thus of order five with characteristics $3^1 3^2 1^3$, the three fundamental elements of the first order coalescing with the single element of the third order at the line at infinity. The sides of the triangle ABC are the three fundamental elements of the second order.

ON THE AXIAL INVOLUTION OF A NET OF CONICS AND ITS ASSOCIATED (1,2) CONNEXES.*

BY A. NARASINGA RAO.

Introduction.

In every net of conics

$$N: \qquad c_1S_1 + c_2S_2 + c_3S_3 = 0$$

where S_1, S_2, and S_3 are ternary quadratics in point or line co-ordinates, and the c's are constants, there is in general a unique member which has a given line u for a principal axis. It follows, therefore, that the two principal axes of conics of the net are co-lines in a Cremona line-involution in the plane. It is shown in this paper that, both in the case of net of conic loci as well as of a net of conic envelopes, there are seven lines in the plane each of which is a principal axis of a linear ∞^1 system of conics of the net, and that the involution of principal axes referred to above, is the Geiser involution defined by these seven lines. It turns out, however, that the seven lines are not in general position, but that six of them are the joins of an orthocentric tetrad of points, the seventh being the line at infinity. The Geiser involution hence degenerates into a Jonquiere involution J of order four, with the line at infinity as a fundamental line of the third order, and the other six lines as fundamental elements of order one.†

A particular case of considerable metrical interest, but one which preserves all the special features of the general case, is that of a net of conics touching the sides of a triangle. The involutlon J now carries each line into the perpendicular bisector of the pair of points on it which are isogonal conjugates with respect to the triangle, or to put it differently, the isogonal pairs on two corresponding lines are cross-pairs (see para 1) of each other. This involution is discussed by synthetic methods in the first two paragraphs, and the equations of the transformation are deduced from geometrical considerations.

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* I take this opportunity of thanking the University of Madras for its research fellowship during which this was one of the investigations carried out.

† Hudson : *Cremona Transformations in Plane and Space*, p. 125.

It is next shown that the transformation J admits of a resolution into three quadratic transformations T_r, which are mutually permutable and together generate an Abelian group of order eight. These transformations are of great metrical interest, and their introduction into the geometry of the triangle should prove of value in investigation in which two of the sides play a role different from the third.

The centres and principal axes of conics of the net N build up, in the terminology of Clebsch, a (1, 2) "Hauptcoincidenz." §§ 5 and 6 give a brief sketch of the main features of the general (1, 2) Hauptcoincidenz, as developed by Clebsch in his "Vorlesungen über Geometrie" (1876), establish the Geiser property, and point out the specialisations projective as well as metrical, which reduce it to the axial coincidence of conics touching three lines. The analytical treatment of the axial coincidence which follows immediately determines the equations of certain incident connexes (connexes containing the coincidence) in certain canonical forms, and leads up to the differential equation of the axial coincidence given in § 8. The solution of this gives a system of curves built up of line elements of the coincidence, there being two through each point, namely, the directions of the two lines associated with the point. When a point x traces one of these curves, the two associated lines u through it envelope a reducible curve which breaks up into the original curve and its evolute. These coincidence-curves are transcendential unless the lengths of the sides of the circumscribing triangle of the conic net are commensurable to one another, and for an equilateral triangle, they form a pencil of class cubics with cusps at I_1, I_2, I_3, the cuspidal tangents being concurrent at I_0.

The detailed treatment which the axial involution of the in-conics of a triangle receives, finds its justification in the fact that it is typical of the general case ; in other words, associated with every net of conic loci or envelopes, there exists a triad of lines such that the principal axes of every conic of the net, are also the principal axes of conics touching the three lines. In the case of a net of conics self-polar with respect to a triangle, these three lines are the sides of the pedal triangle. For conics circumscribing a triangle $A_1A_2A_3$, the three lines form the pedal triangle of the medial triangle of $A_1A_2A_3$.

The last paragraph discusses two cases where the axial involution

degenerates into one of order two, namely (i) when the orthocentric tetrad of the net consists of two cross-pairs, and (ii) when two of the four orthocentric points are at infinity in perpendicular directions.

1. The Axial Involution J of Conics touching three given Lines.

Consider a metrical plane with the Absolute points Ω, Ω'. Then with each pair of points P, P′ we associate another pair Q, Q′—the *cross pair of* P, P′—obtained as the meets of the lines $P\Omega$, $P'\Omega'$ and $P\Omega'$, $P'\Omega$. The relation between a pair and its cross-pair is symmetric, the two pairs and $\Omega\Omega'$ forming the singular members of the same tangential pencil of conics. It follows from this,* that if two of the pairs PP′, QQ′, $\Omega\,\Omega'$ belong to a quadratic Cremona involution in the plane, so does the third pair. But if a quadratic involution interchanges the absolute pair $\Omega\Omega'$, it is an isogonal transformation with respect to the triangle formed by its fundamental points. Hence we have the following result:—

> If a pair of points PP′ are isogonal conjugates with respect to a triangle, so is their cross pair; the lines joining the two pairs are at right angles, and the relation between them is involutoric. ... (1.1)

The involutoric character follows from the fact that each line contains but one pair of a quadratic involution, and from the symmetry of the relation between a pair and its cross-pair.

Now a pair of isogonal conjugates are a pair of conjugate foci of a conic touching the sides of the triangle, the other two foci forming the cross-pair. It follows from this that :—

> The pairs of corresponding lines in the involution J of (1.1) are the principal axes of a conic touching the sides of the fundamental triangle $A_1A_2A_3$. ... (1.2)

The search for the fundamental elements of the transformation is rendered easy by noticing that, since each line corresponds to the perpendicular bisector of the isogonal pair upon it, the correspondence fails only when the isogonal pair on a line is not determinate, or when

* Since pairs such as PP′ are just those which are conjugate with respect to each of a certain pencil of conics.

perpendicularly becomes ambiguous. Hence the fundamental elements are the six bisectors of the angles of the triangle and the line at infinity (the line $\Omega\Omega'$). Each of the former contains an infinity of isogonal pairs, and transforms into any line perpendicular to it, that is, into a pencil of lines; these are, therefore six fundamental elements of the first order.

Again, by (1.1) a pencil of parallel lines is transformed into another pencil of parallel lines (perpendicular to the former), hence the two pencils are projective and the meets of corresponding lines trace a conic through the in- and ex-centres which are fixed points of the isogonal transformation. We thus have the following geometrical construction:

> Corresponding lines of the involution meet on a rectangular hyperbola through the in- and ex-centres, and are parallel to its asymptotes. ... (1.3)

In particular, if a line while remaining parallel to an asymptote of one such rectangular hyperbola moves farther and farther away to infinity, its transform moves into coincidence with the other asymptote. Hence the line at infinity is carried over into an asymptote of one of these hyperbolas, and is thus a fundamental element of the third order and corresponds to the envelope of these asymptotes. This is the Steiner tricusp associated with the in- and ex-centre tetrad.*

Now among the conics through the in- and ex-centres, there are two which touch the line at infinity, the points of contact being Ω and Ω'. Let these two *rectangular parabolas* be denoted by (RP) and (RP'); then it follows from (1.3) that:

---

* The Steiner tricusp of a triangle (the envelope of its pedal lines) is related symmetrically to the vertices and the orthocentre, being equally the envelope of the pedal lines of any of the other three triangles thereof, and is therefore more properly associated with the orthocentric tetrad as a whole. It may be defined more symmetrically either as the double-parabola touching the six joins of the 4 points, or as the envelope of lines on which the three pairs of opposite joins determine segments with a common middle point. That the Steiner tricusp of a triangle "hat drei Ruckkehrpunkte in den Ecken A_i des Urdreiecks" (*Ency. d. Math. Wissenschaften* III, 1, 7 Neure Dreiecksgeometrie S. 1238) is an obvious mistake, and is not true even in the case of an equilateral triangle, since it is symmetrically related to the vertices and the orthocentre and not to the former only.

Every line through Ω or Ω' corresponds in J to itself. The ultimate point of intersection of two such coincident self-corresponding lines lies on one or other of the two rectangular parabolas (RP), (RP'). ... (1.4)

2. The deduction of the actual equations of the transformation now becomes very simple. We note first that the transformation is of the fourth degree, since a pencil of parallel lines is carried over into a curve of class four consisting of the tricusp referred to above, (corresponding to the line at infinity) and a point at infinity (corresponding to the other lines in the pencil). If the lines $(u_1, u_2\ u_3)$ and (u_1', u_2', u_3') correspond, we have equations of the form

$$u_r' = F_r(u_1, u_2, u_3) \qquad (r = 1, 2, 3) \qquad \dots \quad (2.1)$$

where the F's are quartic functions of the ternary variables u_1, u_2, u_3. Here $F_r = 0$ represents the transform of the pencil of lines through the vertex A_r, and this breaks up into three parts. The two bisectors of the angle A_r correspond to two pencils of lines each parallel to the other; on any other line the two isogonal conjugates consist of A_r and the point where the line meets the opposite side, and the transform, which must be equidistant from these two points, envelopes the parabola

$$P_r \equiv u_r^2 - u_s^2 - u_t^2 + 2u_s u_t c_r \qquad (r, s, t = 1, 2, 3) \quad \dots \quad (2.2)$$

having A_r for focus and $A_s A_t$ for directrix, the coordinates being trilinear, and c_r standing for cos A_r. Hence we have three equations of the form

$$F_r = k_r P_r (U_s^2 - U_t^2) \qquad \dots \quad (2.3)$$

where

$$U_1 = u_2 s_3 - u_3 s_2; \quad U_2 = u_3 s_1 - u_1 s_3; \quad U_3 = u_1 s_2 - u_2 s_1;$$

and $s_r = \sin A_r$, the first factor representing the points at infinity on the two bisectors of A_r, and the k's are constants. The ratios $k_1 : k_2 : k_3$ may now be determined from the examination of a particuiar case, say the line $A_2 A_3$ and its ransform which is the line equidistant from A_2 and A_3 It is found that

$$k_1 : k_1 : k_3 = s_2 s_3 : s_3 s_1 : s_1 s_2 \qquad \dots \quad (2.4)$$

Summarising we have :—

The involution J of (1.1) is of order four with the six bisectors of the angles of the triangle as fundamental elements of the first order, (each being the transform of all lines perpendicular to it), and the line at infinity as a fundamental line of the third order corresponding to any tangent to the Steiner tricusp of the in- and ex-centres. The equations of the transformation are given in trilinear co-ordinates by (2.3) and (2.4).

The Three Quadratic Transformations T_r.

3. The quartic Cremona Involution discussed above admits of being broken up, according to Noether's general theorem, into three quadratic involutions. One such resolution is of particular interest and simplicity and deserves study both for its own sake, and for its close connection with the metrical geometry of a triangle.

Consider two point involutions one on each of the bisectors $A_r I_0 I_r$ and $A_0 I_s I_t$ of the angle A_r, which have the common centre A_r and contain respectively the pairs $I_0 I_r$ and $I_s I_t$ Any line cutting the bisectors in P and Q may now be replaced by another passing through the conjugates of P and Q in the two involutions, and the result is a quadratic Cremona line-transformation T_r which has the bisectors of A_r and the line at infinity for fundamental (singular) elements, and interchanges the pair of bisectors of the angles A_s and A_t. The line at infinity (L_∞) being a fundamental element, the transformation carries a pencil of parallel lines into another such pencil, and since in two cases (the bisectors of A_s and A_t) the corresponding lines are perpendicular, we deduce that each line is perpendicular to its transform. The fixed lines of the transformation are the four lines joining the fixed pairs of the two involutions ; and as these are cross-pairs of each other, these lines pass through the absolute points, which form the fixed pair of the involution on the line at infinity.

The transformation T_r carries the pencil of lines through A_s into a parabola touching the bisectors of A_r and A_s, *i.e.*, into the parabola P_t, having A_t for focus and the opposite side for directrix. Again it carries the parabola P_r having A_r for focus and $A_s A_t$ for directrix into a curve of

class three touching the six bisectors, and having the line at infinity for a bitangent, *i.e.*, into the Steiner tricusp of the tetrad of in- and ex-centres.

Consider now the three transformations T_1, T_2 and T_3 associated with the three vertices A_1, A_2, A_3. Since all of them have the line at infinity for a common fundamental line, the product of any two of them say T_r and T_s is a cubic transformation having the bisectors of A_r and A for fundamental lines of the first order, each of these being the transform of *any* line parallel to it, and the line at infinity as a fundamental line of the second order corresponding to every tangent of the parabola P_t. The symmetrical character of the product transformation in the suffixes *r* and *s*, coupled with the fact that a Cremona Transformation with more than three fundamental elements is uniquely determined by its fundamental elements and their transformed loci*, leads us to conclude that $T_rT_s \equiv T_sT_r$. Thus any two of the transformations T_1, T_2, T_3 are commutative, and they generate an Abelian group G_8 of order eight, the product of all the transformations being in fact the Jonquiere transformation considered at the beginning of the paper.

To prove the last statement, we notice firstly that since the three commutative quadratic involutions have for singular elements the bisectors and L_∞, their product is a quartic involution with the six bisectors for singular lines of the first order, and L_∞ for one of the third order. Each of the former is carried over into an arbitrary line perpendicular to it, while the line L_∞ is carried over by T_1, T_2 and T_3 successively into the pencil of lines through A_1, the parabola P_3 with A_3 for focus and A_1A_2 for directrix, and finally into the Steiner tricusp of $I_0I_1I_2I_3$. The transformations T_r, T_s, T_t (in any order) and J have thus the same fundamental elements and the same transformed loci, and must hence be identical. We have thus shown that:

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* If T and T′ are two Cremona transformations having the same fundamental elements and associated loci, the product of either by the inverse of the other is a Cremona transformation without any fundamental elements, *i.e.*, a collineation. Each of the common fundamental elements is a fixed element of this collineation, and as there are at least four of them, it must be the identical collineation leaving every point in the plane invariant. Hence $T \equiv T'$.

A metrical triangle determines three quadratic line involutions T_1, T_2, T_3, namely, those whose fundamental elements are the two bisectors of any angle and the line at infinity; each of these interchanges the bisectors of the other two angles, and replaces each line by one in a perpendicular direction. The three involutions are commutative, and their continued product is the quartic involution J. ... (3.1)

4. The involution of lines defined by T_r admits of being thrown into other forms possessing greater geometrical significance.

Let Q_r denote the unique in-conic of the triangle $A_1A_2A_3$ having A_r for centre. Its principal axes are then a pair of lines through A_r which correspond in J and are hence the two bisectors of A_r, and its foci are the two pairs of isogonal conjugates on these lines which are equidistant from A_r. These are, in fact, the fixed points of the involution determined by T_r on these two bisectors*. Pairs of corresponding lines in T_r meet each bisector of A_r in pairs of points separating harmonically the focal pair on that line, and hence constitute degenerate order conics apolar to the two class conics consisting of the focal pairs on each bisector. It follows that the order conics are also apolar to every member of the tangential pencil determined by the two point pairs, *i.e.*, to every conic confocal with Q_r. It follows from this that :—

The tangents to the two confocals to Q_r which pass through any given point are corresponding lines in the involution T_r; that is, each line is carried over by T_r into the normal to that confocal to Q_r which touches it. ... (4.1)

From the permutability of the transformations T_1, T_2, T_3 it follows that any line *u* is carried over by the group G_8 into two sets of four lines,

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* The configuration formed by the 12 foci of the conics Q_r possesses some remarkable properties. They lie in pairs on the six bisectors; those lying on three non-concurrent bisectors lie on a circle with respect to which the other six form three pairs of inverses. The four circles thus obtained are mutually orthogonal circles with their centres at I_0, I_1, I_2, I_3. The 12 points form thus the total intersection system of the above four circles. The pair of points on any bisector is carried into itself by inversion in any of the four circles.

the lines of each set being parallel and perpendicular to those in the other set. The lines u, u_{12} $[= T_1T_2(u) = T_2T_1(u)]$, u_{23}, u_{31} form one set of lines, while u_1 $(= T_1u)$, u_2, u_3 and u_{123} form the other. It follows from (4.1) that, of the sixteen pairs of lines obtained by taking one from each set, all except four are tangents to two confocals to one of the conics Q_1, Q_2 or Q_3 through their intersection. Thus the pairs of lines u_0u_r, $u_{rs}u_s$, $u_{rst}u_{st}$ are tangents to the two confocals to Q_r which pass through their intersection. The four exceptional pairs u_0u_{123}, u_1u_{23}, u_2u_{31}, u_3u_{12} are corresponding lines in $J \equiv T_1 . T_2 . T_3$.

The Connex of Clebsch and the (1, 2) principal coincidence.

5. The axial involution J associates with each line in general position a definite point on it, namely, its intersection with its transform. This point is, in fact, the mid-point of the isogonal pair on the line, and may be called its *central point.* The central point of a pair of corresponding lines is the same, and is the centre of the conic having these lines for principal axes. When the central point X of a line is given, the line itself is determined in one of two ways, for the involution J carries the totality of lines through X into tangents to a class quartic, and the four common lines belonging to both the systems form an invariant tetrad for J. But from (1.4) every line through Ω or Ω' is an invariant line, and hence the four lines mentioned above consist of $X\Omega$, $X\Omega'$, and a pair of corresponding lines. Rejecting the pair $X\Omega$, $X\Omega'$ we obtain a single pair of properly corresponding lines through each point X in the plane. We now proceed to consider the relation between a line and its central point.

A *connex* according to Clebsch* is a form

$$f = (\overset{m}{x_1, x_2, x_3}; \overset{n}{u_1, u_2, u_3}) = a_x^{\,m} u_\alpha^{\,n} \qquad \dots \quad (5.1)$$

in two sets of contragredient variables x and u, and homogeneous in each set. If these variables be interpreted as the co-ordinates of a point and a line in the projective plane, the connex (5.1) associates with each point x in the plane a class curve K_n, and with each line u a point locus C_m. The common elements of two connexes is a *coincidence*, and in particular the common elements of any connex and the

* Clebsch : *Vorlesungen über Geometrie* (1876), S.936.

principal connex (Hauptkonnex)

$$u_1x_1 + u_2x_2 + u_3x_3 = 0 \quad \ldots \quad \ldots \quad (5.2)$$

is its *principal coincidence* (Hauptkoincidenz). The principal coincidence of the (m, n) connex (5.1) associates with each point, n lines through it, and with each line m points on the line. We may thus restate our result in the form:

The totality of lines in the plane each associated with its central point forms an (1, 2) principal coincidence. ... (5.3)

The (1, 2) principal coincidence

$$a_x u_\alpha^2 \equiv b_x u_\beta^2 \equiv c_x u_\gamma^2 \ldots = 0 \qquad u_x = 0 \quad \ldots \quad (5.4)$$

associates with the line u, the point whose co-ordinates are $x_r = u_\alpha^2 (au)_{st}$ $(r, s, t = 1, 2, 3)$. Now the three class cubics

$$u_\alpha^2 (au)_{st} = 0 \quad \ldots \quad (5.5)$$

have seven tangents in common and these will be *singular lines* of the coincidence since for these, the co-ordinates of the associated point are indeterminate, and accordingly x may be taken to be *any* point in the line.

Again, as a point x traces a line v, we see from (5.5) that the two coincidence-lines u through it envelope the class cubic

$$K_v = (auv)\, u_\alpha^2 \quad \ldots \quad (5.6)$$

the *isologue* of v in the Jonquiere involution J. It is clear from its equation that the curve K_v belongs to the net (5.5) and hence touches the seven singular lines of the coincidence. If v is one of a pencil of lines, we have an infinity of curves K_v, one associated with each line of the pencil, and all these curves touch the seven singular lines and the pair of coincidence lines through the vertex of the pencil. It follows therefore that:

The seven singular lines of a coincidence determine it completely, ... (5.7)

for given any line u, there exists in general a unique conjugate line u' such that u, u', and the seven singular lines fail to determine a cubic uniquely; the two lines u and u' are then the coincidence lines through their intersection.

It is thus clear that:—

The involution of lines determined by a general (1, 2) coincidence is the Geiser involution of its seven singular lines, that is, it is the involution of pairs of lines which with the seven singular lines of the coincidence fail to determine a class cubic uniquely. ... (5.8)

The Geiser involution is, in general, of order eight. In fact, if we consider a pencil of lines vertex y, the locus of the point x associated by the (1, 2) coincidence with each line is an order cubic:

$$C_3 = a_x (\alpha x y)^2 = 0 \qquad \dots \quad (5.9)$$

As x traces this cubic, the pair of lines u through it envelopes a class 9-ic which we obtain by replacing x by $u^2_\alpha\,(au)$ in (5.8); but this breaks up into the factor u_y corresponding to the pencil of lines through y, and a class 8-ic

$$(abu)\, u^2_\beta\, u^2_\gamma\, u^2_\delta\ \{ u_y\, c_\alpha\, d_\alpha - 2u_\alpha\, d_\alpha\, c_y \} = 0 \quad \dots \quad (5.10)$$

which is the proper homologue of the point y. The seven singular lines are seen to be triple tangents of (5.10), and are thus fundamental elements of the third order. (HUDSON: *Cremona Transformations*, p. 124.)

When three of the seven lines are concurrent at a point z, (5.10) throws out a factor u_z since a proper curve of class eight cannot have three concurrent triple tangents, and the order of the involution reduces by unity. Thus when the seven singular lines are the sides and diagonals of a quadrilateral, the order decreases by five and reduces to a quadratic transformation. When the two coincidence lines through a point x are the principal axes of the in-conic of a triangle which has its centre at x, the singular lines are, as we have already seen, the six bisectors of the angles of the triangle and the line at infinity. The involution reduces to one of order four, and is in fact the Jonquiere involution J. Remembering (5.7), that the seven singular lines of a (1, 2) coincidence determine it uniquely we may state:

The axial coincidence of the in-conics of a triangle exhibits all the projective features of a principal coincidence six of whose lines are the sides of a complete quadrangle; a projective transformation carrying the seventh line into the line at infinity and the quadrangle into an ortho-centric quadrangle, converts the latter into the former. ... (5.11)

6. We now proceed to discuss the peculiar features which this specialisation brings about.

When the point x lies on the associated connex curve $a_x u_\alpha{}^2 = 0$ the two connex lines through the point become identical. Hence the locus of points for which the two lines of the principal coincidence fall together, is the quartic $F = a_x\, b_x\, (\alpha\beta x)^2 = 0$, which in the general case is unspecialised. In the present case we have seen (1.4) that this locus F breaks up into the two rectangular parabolas (RP) and (RP′) through $I_0 I_1 I_2 I_3$. Again the envelope of the coincident line pairs u as the point x moves along F is a class sextic* $\phi = (ubu)\, c_\alpha\, u_\alpha\, u^2_\beta\, u^2_\gamma = 0$. In the case of the axial coincidence we are considering, as the point moves along (RP) and (RP′) it throws off the pencils with vertices I_0. I_1, I_2, I_3 as it passes these positions (since the axial pair through any of these points is indeterminate), and the envelope proper consists of the two absolute points Ω and Ω', one corresponding to (RP) and the other to (RP′). The sextic ϕ thus breaks up into the six pencils whose vertices are the absolute points and the in- and ex-centres.

The axial (1, 2) coincidence derived from the quartic involution J may be contrasted with one derived from a quadratic involution, say the involution T_r of § 3. Here the (1, 2) coincidence associates with each line the point in which it is met by its transform, and with each point the pair of corresponding lines of T_r which pass through it. The lines on which the associated point is indeterminate, consist of the three singular lines of T_r whose transforms are indeterminate, and the four self-corresponding lines. But these seven lines are now the sides and diagonals of the quadrilateral formed by the four fixed lines of the transformation. The locus F of points for which the two associated lines fall together breaks up into the four fixed lines of the transformation, indicating a further stage of degeneracy than in the previous case, while the class sextic ϕ breaks up as before into six pencils of lines whose vertices are the absolute points and the foci of Q_r. Hence

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* Clebsch : *loc. cit.*

for the (1, 2) coincidence of a quadratic involution, the seven singular lines are the sides and diagonals of a quadrilateral, the F-quartic breaks up into the four sides and the ϕ sextic into six points. The six points are the intersections of the four sides. ... (6.1)

7. The Axial Coincidence and its Incident Connexes.

The actual equations of the axial coincidence may be obtained either from the equations for J given in § 2, or directly from the property that x is the mid-point of the isogonal pair on the line u. For trilinear co-ordinates, these equations are given by :—

$$x_1 : x_2 : x_3 =$$

$$\frac{S_0}{U_0}+\frac{S_1}{U_1}-\frac{S_2}{U_2}-\frac{S_3}{U_3} : \frac{S_0}{U_0}-\frac{S_1}{U_1}+\frac{S_2}{U_2}-\frac{S_3}{U_3} : \frac{S_0}{U_0}-\frac{S_1}{U_1}-\frac{S_2}{U_2}+\frac{S_3}{U_3} \qquad \dots (7.1)$$

where

$$\left.\begin{array}{ll} S_0 = s_1 + s_2 + s_3; & S_r = s_r - s_s - s_t \\ U_0 = u_1 + u_2 + u_3 & U_r = u_r - u_s - u_t \end{array}\right\} \quad (r, s, t = 1, 2, 3) \dots \quad (7.2)$$

so that,

$$S_0 + S_1 + S_2 + S_3 = 0 = U_0 + U_1 + U_2 + U_3. \qquad \dots (7.3)$$

These equations lead immediately to

$$u_1 x_1 + u_2 x_2 + u_3 x_3 = 0, \qquad \dots (7.4)$$

showing that the coincidence is a principal one. Any lineo-quadratic relation between x and u in (7.1) would give the equation of a (1, 2) connex which together with the principal connex (7.4) defines the coincidence. When one such connex $a_x u^2_\alpha$ has been determined, all the quadratic connexes through the given coincidence are given by $a_x u_\alpha^2 + u_x u_\beta$ where u_β is an arbitrary linear form in u. We may however definitize the connex giving it the special shape

$$a_x v_1 v_2 + b_x v_2 v_3 + c_x v_3 v_1 = 0 \qquad \dots (7.5)$$

where the v's are linear combinations of the u's. The geometrical significance of the form (7.5) is that the conics associated with all points x in the plane are inscribed in the same triangle, namely, the one whose vertices are $v_r = 0$. When the line u is a side of this triangle, all the

co-efficients in (7.5) vanish, and the corresponding point x is indeterminate. Hence a necessary condition for the possibility of the form (7.5) is that the three lines v_r be singular lines of the coincidence, and it may be shown that this condition is also sufficient. The reduction to the canonical form is thus possible in $7.6.5/1.2.3 = 35$ ways. We proceed to obtain the equations of (1,2) connexes through the axial coincidence in the canonical from (7.5).

From the equations (7.1) we have immediately:

$$\left.\begin{aligned} \tfrac{1}{2}\, U_0\, U_r\, (x_s + x_t) &= S_0\, U_r - S_r\, U_0 \\ \tfrac{1}{2}\, U_s\, U_t\, (x_s - x_t) &= S_s\, U_t - S_t\, U_s \end{aligned}\right\} \quad \dots \quad (7.6)$$

whence it follows that

$$S_1\,(x_2 - x_3)\, U_2\, U_3 + S_2\,(x_3 - x_1)\, U_3\, U_1 + S_3\,(x_1 - x_2)\, U_1\, U_2 = 0 \quad \dots \quad (7.7)$$

$$S_0\,(x_s - x_t)\, U_s\, U_t + S_t\,(x_r + x_t)\, U_0 U_s - S_s\,(x_r + x_s)\, U_0\, U_t = 0 \quad \dots \quad (7.8r)$$

giving the equations of four of the connexes through the axial coincidence corresponding respectively to the triangles $I_1 I_2 I_3$ and $I_0 I_s I_t$. Any other (1, 2) connex through the coincidence may be obtained as a linear combination of (7.7) and (7.8).

8. The Differential Equation of the Coincidence.

Any (1, 2) connex associates with each point in the plane, a pair of lines through it, and hence determines two line-elements through each point. If $x + dx$ be a neighbouring point in one of these directions, we may write $u_r = (x\ dx)_{st}$ in the equation (5.4) of the connex, and obtain the differential equation $a_x\,(x\ dx\ \alpha)^2 = 0$. Dually we write $x_r = (u\,du)_{st}$ and obtain also the dual form $(a u du)\ u_\alpha{}^2 = 0$. Either of these when solved gives us the *curves of the coincidence, i.e.*, a system of curves two through each point in the plane, the tangents to which at any point are the coincidence lines through it.

Replacing x by $(u\ du)$ in (7.7) we obtain the differential equation

$$\sum \frac{S_1}{U_1}\,(u_3 du_1 - u_1 du_3 - u_1 du_2 - u_2 du_1) = 0,$$

which on replacing the u's in terms of U's, takes the shape:

$$\sum dU_1 \left\{ \frac{S_1}{U_1} - \frac{S_0}{U_0} \right\} = 0$$

or

$$\frac{dU_1}{U_1} S_1 + \frac{dU_2}{U_2} S_2 + \frac{dU_3}{U_3} S_3 + \frac{dU_0}{U_0} S_0 = 0$$

by (7.3). The integral curves* are thus given by

$$U_0^{S_0} U_1^{S_1} U_2^{S_2} U_3^{S_3} = \text{constant},$$

which in virtue of (7.3) is homogeneous and of degree zero in u on both sides.

The curves are transcendental unless the ratio of the lengths of the sides are all rational numbers. In the latter case, if for a real triangle a, b, c be the least integers proportional to the lengths of the three sides, the curves are of class $a + b + c$ and may be written

$$(u_2 + u_3 - u_1)^{b+c-a} (u_3 + u_1 - u_2)^{c+a-b} (u_1 + u_2 - u_3)^{a+b-c}$$
$$= \text{const.} \times (u_1 + u_2 + u_3)^{a+b+c}. \quad \dots \quad (8.1)$$

Since $a + b > c$, etc., the lines I_0I_1, I_0I_2, and I_0I_3 are all multiple tangents of order $b + c - a$, $c + a - b$, and $a + b - c$ respectively, the points of contact all coalescing at I_1, I_2 and I_3.

When $a = b = c$ so that the triangle is equilateral, the curves (8.1) take the shape

$$(u_2 + u_3 - u_1) (u_3 + u_1 - u_2) (u_1 + u_2 - u_3) = \text{const.} \times (u_1 + u_2 + u_3)^3. \quad \dots \quad (8.2)$$

and represent a pencil of curves with cusps at I_1, I_2 and I_3 the cuspidal tangents being respectively I_1I_0, I_2I_0, and I_3I_0.

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* The integration of the differential equation of a general (1,2) coincidence, *i.e.*, the determination of all curves with characteristics $\mu = 2$, $\nu = 1$, has been discussed by Darboux, and Fouret *Comptes Rendus* LXXXVI and A. Legouse, *Thèse* Paris 1878. None of these papers were, however, available for consultation.

From the defining property of a (1,2) coincidence, the curves (8.1) have the characteristics $\mu = 2$, $\nu = 1$, *i.e.*, one curve of the system touches any line u, and two pass through a point x. In the former case the point of contact is the point x associated with the line u, while the two lines associated with a point x are the tangents to the two curves through it. In particular we may conclude that the curves (8.1) form an orthogonal system.

On the other hand when the coincidence is one derived from a quadratic Cremona Transformation, the corresponding curves are the system of conics touching the four fixed lines of the transformation. Thus for the transformation T_r of § 3, these conics are confocal conics; this is the real content of the statement (4 1).

The curves of a coincidence may also be viewed from another standpoint. When the point x in the plane describes any curve $v_x^n = 0$ of order n, the two coincidence lines through x envelope a class $3n$-ic,

$$(auv)\,(buv)\ \ldots\ldots\ u_\alpha^2\ u_\beta^2\ \ldots\ldots = 0,$$

which is in general algebraically irresoluble; but when the curve satisfies the differential equation of the coincidence, the envelope breaks up into the original curve itself, and another of class $2n$, which is the envelope of the other coincidence line through the point. From the property that the two lines are at right angles, it follows that the second curve is the evolute of the first.

9. The Axial Coincidence of a general net.

The somewhat detailed discussion which the axial coincidence of the inconics of a triangle has received so far, is justified by the fact that it is typical of the axial coincidence of a general net of conics; in fact, we shall now show that in the case of any general net, the relation between the principal axes is precisely that of the principal axes of inconics of a certain triangle defined by the net.

Firstly, we notice that the statement that a point y and a line v are pole and polar with respect to a conic, presents two linear conditions to a conic locus $a_x^2 = 0$, or to a conic envelope $u_\alpha^2 = 0$, namely $a_y a_x \equiv v_x$ or $u_\alpha v_\alpha \equiv u_y$. Hence if we take a net

$$N: \quad c_1 S_1 + c_2 S_2 + c_3 S_3 = 0 \qquad \ldots \quad (9.1)$$

of conic loci or envelopes, it follows that

> there is, in general, a unique member of (9.1) which has a given point y for centre or a given line v for principal axis, ... (9.2)

for in the former case we take v to be the line at infinity and in the latter, y as the point at infinity in the direction perpendicular to that of v, and apply the condition of polarity of y and v. It follows also from the linear character of the condition, that if we have two conics of the net which have the line v for a principal axis, a like property holds for all members of the pencil defined by them.

From (9.2) it is clear that the centres and principal axes of conics of the net build up a (1,2) coincidence, for through each point pass two principal axis, and on each principal axis there is a unique centre. Consequently, the relation between the axes of the same conic is by (5.8) the Geiser involution determined by the seven singular lines of the coincidence. One of these lines is the line at infinity, for when the line at infinity is a principal axis, the conic becomes a parabola and this presents a single condition to the net N so that the conic fails to be determined uniquely. We shall now show that

> the seven singular lines of the axial (1,2) coincidence of the general net N consist of the line at infinity and the six joins of an orthocentric tetrad of points. ... (9.3)

Consider first

Case (i) where N is a tangential net of conics,

so that the S's in (9.1) are ternary quadratics in u. The condition that a member of the net pass through either circular point is quadratic in the co-efficients c; hence there are four circles in the net. Let $u_\alpha = 0$, $u_\beta = 0$, $u_\gamma = 0$ be the equations of the centres of three of the circles. Then the equations of the circles are of the form $u_\omega^2 + ku_\alpha^2 = 0$, etc., where $u_\omega^2 = 0$ is the equation of the absolute points so that the net itself is given by

$$u_\omega^2 (c_1 + c_2 + c_3) + c_1 k_1 u_\alpha^2 + c_2 k_2 u_\beta^2 + c_3 k_3 u_\gamma^2 = 0 \quad \dots \quad (9.4)$$

If we ignore the first term on the left in (9.4), the others represent the self-polar net of the triangle formed by the points α, β, γ, and this contains

one circle, namely, the polar circle of the triangle. The effect of the first term is to replace this circle by another concentric with it, whence it follows that the centres of the four circles in the net form an orthocentric tetrad.

Now the line joining the centres of two of the circles is a principal axis of both, and therefore also of the tangential pencil defined by them. The six joins of the centres are thus all singular lines of the axial coincidence of the net, and if we denote by A_1, A_2, A_3 the intersection of the lines in pairs, the axial involution is precisely that of conics inscribed to the triangle $A_1A_2A_3$.

Next we consider

Case (ii) when N is a net of conic loci.

The locus of double points of members of the net is the Jacobian of the net, which is a cubic curve. Corresponding to the three intersections of the cubic with the line at infinity, we have three conics each consisting of a pair of parallel lines. In each case take a line parallel to the two lines and equidistant from both. Let $u_x = 0$, $v_x = 0$, $w_x = 0$ be the three lines thus obtained, so that the equations of the three parallel line-pairs are $l_x^2 + k_1 u_x^2 = 0$, etc., where $l_x = 0$ represents the line at infinity, and the net takes the shape:

$$(c_1 + c_2 + c_3)\, l_x^2 + k_1c_1u_x^2 + k_2c_2v_x^2 + k_3c_3w_x^2 = 0 \; \ldots \quad (9.5)$$

Clearly, the lines $u_x = 0$, $v_x = 0$, $w_x = 0$ are singular lines of the axial coincidence, since the centre of the conic having any of these for a principal axis, is indeterminate and may be any point on the line.

Again when $c_1 = 0$ (9.5) gives a pencil of conics all of which have a common centre at the point $v_x = 0$, $w_x = 0$. As the conic varies the principal axes of members of the pencil exhaust all pairs of perpendicular lines through the point; hence there is a conic of the pencil which has for a principal axis, the perpendicular from the point $v_x = 0$, $w_x = 0$ on the line $u_x = 0$. But there is also another conic having this line for a principal axis, namely, the unique circle included in the net. This circle has its centre at the orthocentre of the triangle formed by the three lines $u_x = 0$, $v_x = 0$, $w_x = 0$; for equation (9.5) shows that it is concentric with the polar circle of the triangle.

Since there are two conics which have the same line for principal axis, this property holds for all conics of the pencil determined by them, and as the centres of the two conics are different, every point on the joining line will be a centre of some conic of the pencil, and therefore also of the net. Hence the vertices and orthocentre of the triangle form four points whose six joins are singular lines of the axial coincidence of the net, the seventh being the line at infinity.

It follows, as in the case of the tangential net, that the axial involution is the same as that of conics touching the sides of the pedal triangle of that formed by $u_x = 0$, $v_x = 0$, $w_x = 0$.

A particular case of interest is that of conics circumscribing a triangle. Here the three parallel pairs are formed by each side and a parallel to it through the opposite vertex, and the lines $u_x = 0$, $v_x = 0$, $w_x = 0$, are the lines joining the mid-points of the sides. The orthocentric tetrad consists of the three mid-points and the circum-centre.

For a net of conics self-polar with respect to a triangle, the parallel pairs are the three sides each taken twice, and the singular lines are the sides and perpendiculars of the common self-polar triangle, and the line at infinity.

10. The degeneracy of the axial involution.

While the axial involution of a net of conics is ordinarily a Jonquiere quartic involution, any special relation in which the orthocentric tetrad stands to the line at infinity, may reduce it to one of lower order.

Thus if the ortho-centric tetrad consists of two cross pairs LL′, MM′ the seven singular lines of the axial coincidence are the sides and diagonals of a quadrilateral, and by (5.7) and (6.1) it follows that the involution is of order two with the joins of the two pairs and the line at infinity for fundamental lines. The involution will be identical with T_r discussed in § 3, if the two cross-pairs are the two pairs of conjugate foci of Q_r.

To find a net of conics for which this happens, take L, L′ as the points $(\pm\ \alpha, 0)$ so that M, M′ are $(0, \pm \alpha i)$, the axes being rectangular. The tangential net of conics

$$o_1 (m^2 \alpha^2 - n^2) + o_2\, mn + o_3\, nl = 0$$

where $lx + my + n = 0$ is the cartesian equation of a straight line, includes four circles of radii a, a, 0, 0 with centres at L, L′, M, M′ respectively and is by the argument in § 9, a net of the kind required. It consists of conics touching a line MM′ at the mid-point O, and for which the tangents from M and M′ are parallel.

Examples of point-nets for which the axial involution is quadratic, owing to the same reason, are furnished by

(i) conics self-polar with respect to a triangle two of whose sides are isotropic lines, and

(ii) conics circumscribing such a triangle.

Another important case in which the axial involution reduces to one of order two, is that in which two points of the orthocentric tetrad are at infinity in perpendicular directions. This happens, for instance, in the case of conics touching three lines of which two are parallel. That the axial involution is quadratic, is here clear from geometrical considerations, for the centres of all such conics lie on a line l parallel to and midway between the two parallel sides. Also there are an infinity of conics of the net with any given point on this line for centre, and the principal axes of these exhaust all lines through the point. Hence the axial involution is here a quadratic "central" involution in which corresponding lines meet on l and are at right angles.

The mere coincidence of two or even three points of the ortho-centric tetrad, does not bring out any degeneracy in the axial involution which still remains of order four, the conditions of contact with the six lines undergoing the necessary modifications due to the coincidence of the lines.

ON THE EQUILIBRIUM OF A THICK ELASTIC PLATE ONE OF WHOSE FACES IS SUBJECTED TO A GIVEN DISTRIBUTION OF PRESSURE, AND THE APPLICATION OF THE ABOVE TO THE CASE OF A CIRCULAR PLATE.*

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The problem of the determination of stress and strain in a thick elastic plate one of whose faces is subjected to a given distribution of pressure, has been solved for two cases only, *viz.*, when the pressure is uniform, or when it varies uniformly with one of the co-ordinates. The present paper consists of two parts. In the first part, the general solutions of the displacement and stress-components have been obtained when one of the faces of the plate is subjected to a pressure which varies according to the law $p = p_0 xy$, where p_0 is a constant. It is found that the formulæ for the stress-components calculated from the results do not exactly agree with the corresponding formulæ of the approximate theory of Kirchhoff, but that the latter formulæ are approximately correct for the assumed distribution of pressure when the thickness of the plate is sufficiently small. Thus the approximate theory is further confirmed by the results of this part.

In the second part, the results obtained in the first part have been utilised to determine the exact solutions of the displacement and stress at any point in a circular plate with clamped edge, and subjected to the assumed kind of pressure distribution.

My best thanks are due to Dr. N. M. Basu, D.Sc., who suggested this problem to me, and under whose guidance I carried on the investigation.

2. Since the only external force is a variable normal pressure over one of the faces of the plate, the problem is one of the equilibrium of an isotropic elastic body under surface tractions only.

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* An abstract of this paper was read at the Indian Science Congress, 1928.

The stress equations of equilibrium are of the type

$$\frac{\partial X_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial X_z}{\partial z} = 0 \qquad \ldots \quad (1)^*$$

and we know that the stress-components and θ satisfy equations of the following type

$$\nabla^2 X_z = -\frac{1}{1+\sigma}\frac{\partial^2 \theta}{\partial x^2} \qquad \ldots \quad (2)$$

and

$$\nabla^2 Y_z = -\frac{1}{1+\sigma}\frac{\partial^2 \theta}{\partial y \partial z} \qquad \ldots \quad (3)$$

where

$$\theta = X_x + Y_y + Z_z \qquad \ldots \quad (4)$$

and ∇^2 stands for $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$.

We know that θ is a harmonic function, *i.e.*, it satisfies the equation $\nabla^2\theta = 0$, at all points within a body held strained by surface tractions only, and that for such a body at any point within it, each of the stress-components satisfies the equation $\nabla^4\phi = 0$.

Since the only external force is a variable normal pressure over a face $z = h$, say, we have

$Z_z = -p_0 xy$ over $z = h$

$Z_z = 0$ over $z = -h$

$X_z = Y_z = 0$ over both the faces $z = h$ and $z = -h$.

Also the equations of equilibrium gives $\frac{\partial Z_z}{\partial z} = 0$ over $z = h$ and $z = -h$.

3. *To determine* Z_z.

Z_z satisfies the following conditions :

$\nabla^4 Z_z = 0$ at all points within the body ;

$\frac{\partial Z_z}{\partial z} = 0$ over the faces $z = h$, $z = -h$;

* Throughout this paper Love's notation has been used.

$$Z_z = -p_0xy \text{ at } z = h\,;\; Z_z = 0 \text{ at } z = -h.$$

A particular solution is given by

$$Z_z = \tfrac{1}{4}\, p_0h^{-3}xy\,(z^3 - 3\,h^2z - 2\,h^3) \qquad \dots \quad (5)$$

(ii) *To 'etermine* θ.

We have

$$\frac{\partial^2\theta}{\partial z^2} = -\tfrac{3}{2}\,(1+\sigma)\,p_0h^{-3}xyz \qquad \dots \quad (6)$$

Also

$$\nabla^2\theta = 0. \qquad \dots \quad (7)$$

The most general solution of these equations may be easily thrown into the form

$$\theta = -\tfrac{1}{4}\,(1+\sigma)\,p_0h^{-3}xyz^3 + \tfrac{1}{8}\,(1+\sigma)\,p_0h^{-3}xyz\,(x^2+y^2) + z\theta_1 + \theta_0 \qquad \dots \quad (8)$$

where θ_0 and θ_1 are two arbitrary functions satisfying equations of the type $\nabla_1^2 f = 0$ where ∇_1^2 stands for $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. Let us for the present, neglect the terms $z\theta_1$ and θ_0. The stress-components arising out of these terms are well-known and they may be easily utilised in solving particular problems. It will therefore serve our purpose if we take, at present,

$$\theta = -\tfrac{1}{4}\,(1+\sigma)\,p_0h^{-3}xyz^3 + \tfrac{1}{8}\,(1+\sigma)\,p_0h^{-3}xyz\,(x^2+y^2) \qquad \dots \quad (9)$$

(iii) *To determine* X_z *and* Y_z.

X_z and Y_z are found to satisfy the following equations :

$$\frac{\partial X_z}{\partial x} + \frac{\partial Y_z}{\partial y} + \frac{3}{4}\,p_0h^{-3}\,xy\,(z^2 - h^2) = 0$$

$$\nabla^2 X_z = -\frac{1}{1+\sigma}\,\frac{\partial^2\theta}{\partial x\partial z} = -\tfrac{1}{8}\,p_0h^{-3}\,y\,(3x^2 + y^2 - 6z^2)$$

$$\nabla^2 Y_z = -\frac{1}{1+\sigma}\,\frac{\partial^2\theta}{\partial y\partial z} = -\tfrac{1}{8}\,p_0h^{-3}\,x\,(x^2 + 3y^2 - 6z^2).$$

They also satisfy the following boundary conditions :

$$X_z = 0 \text{ over } z = h \text{ and } z = -h$$

$$Y_z = 0 \text{ over } z = h \text{ and } z = -h.$$

It is easy to show that a particular solution of the equations satisfying these conditions is given by

$$X_z = \tfrac{1}{16} p_0 h^{-3} y (h^2 - z^2)(3x^2 + y^2) + \tfrac{1}{8} p_0 h^{-3} y (h^2 - z^2)(2h^2 - z^2) \quad \ldots \quad (10)$$

and

$$Y_z = \tfrac{1}{16} p_0 h^{-3} x (h^2 - z^2)(x^2 + 3y^2) + \tfrac{1}{8} p_0 h^{-3} x (h^2 - z^2)(2h^2 - z^2) \quad \ldots \quad (11)$$

(iv) *To determine* X_x, Y_y, X_y.

It is easy to show that X_x, Y_y, X_y satisfy the following equations:

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} = \tfrac{1}{8} p_0 h^{-3} yz (3x^2 + y^2 - 4z^2 + 6h^2) \quad \ldots \quad (12)$$

$$\frac{\partial X_y}{\partial x} + \frac{\partial Y_y}{\partial y} = \tfrac{1}{8} p_0 h^{-3} zx (x^2 + 3y^2 - 4z^2 + 6h^2) \quad \ldots \quad (13)$$

Also, we have

$$\nabla^2 X_x = -\frac{1}{1+\sigma}\frac{\partial^2 \theta}{\partial x^2} = -\tfrac{3}{4} p_0 h^{-3} xyz \quad \ldots \quad (14)$$

$$\nabla^2 Y_y = -\frac{1}{1+\sigma}\frac{\partial^2 \theta}{\partial y^2} = -\tfrac{3}{4} p_0 h^{-3} xyz \quad \ldots \quad (15)$$

$$\nabla^2 X_y = -\frac{1}{1+\sigma}\frac{\partial^2 \theta}{\partial x \partial y} = -\tfrac{1}{8} p_0 h^{-3} z (3x^2 + 3y^2 - 2z^2) \quad \ldots \quad (16)$$

and

$$X_x + Y_y = \theta - Z_z$$
$$= \tfrac{1}{4} p_0 h^{-3} xy \left[-(2+\sigma) z^3 + z\left\{\tfrac{1}{2}(1+\sigma)(x^2+y^2) + 3h^2\right\} + 2h^3\right] \quad \ldots \quad (17)$$

The equations (12) and (13) are evidently satisfied by the following forms for X_x, Y_y and X_y. They are:

$$X_x = \tfrac{1}{4} p_0 h^{-3} xyz (3h^2 - 2z^2) + \tfrac{1}{8} p_0 h^{-3} xyz (x^2 + y^2) + \frac{\partial^2 \chi}{\partial y^2} \quad \ldots \quad (18)$$

$$Y_y = \tfrac{1}{4} p_0 h^{-3} xyz (3h^2 - 2z^2) + \tfrac{1}{8} p_0 h^{-3} xyz (x^2 + y^2) + \frac{\partial^2 \chi}{\partial x^2} \quad \ldots \quad (19)$$

$$X_y = -\frac{\partial^2 \chi}{\partial x \partial y} \quad \ldots \quad (20)$$

where χ satisfies the equation :

$$\nabla_1{}^2\chi = p_0 h^{-3} xy\left[\frac{2-\sigma}{4}z^3 - \frac{1-\sigma}{8}(x^2+y^2)z - \tfrac{3}{4}h^2 z + \tfrac{1}{2}h^3\right] \quad \dots \quad (21)$$

We may easily show that the equations (14), (15) and (16) reduce to :—

$$\frac{\partial^2}{\partial y^2}\left[\nabla^2\chi - \tfrac{1}{8}p_0 h^{-3} xyz(x^2+y^2-2z^2)\right] = 0; \quad \dots \quad (22)$$

$$\frac{\partial^2}{\partial x^2}\left[\nabla^2\chi - \tfrac{1}{8}p_0 h^{-3} xyz(x^2+y^2-2z^2)\right] = 0; \quad \dots \quad (23)$$

and

$$\frac{\partial^2}{\partial x\partial y}\left[\nabla^2\chi - \tfrac{1}{8}p_0 h^{-3} xyz(3x^2+3y^2-2z^2)\right] = 0 \quad \dots \quad (24)$$

Equations (22), (23), (24) clearly show that the expression

$$\nabla^2\chi - \tfrac{1}{8}p_0 h^{-2} xyz(3x^2+3y^2-2z^2)$$

is a linear function of x and y. The expressions for X_x, Y_y and X_y remain unaffected if we regard this expression as equal to zero, and then χ is found to satisfy the equation :

$$\frac{\partial^2\chi}{\partial z^2} = -p_0 h^{-3} xy\left[\frac{3-\sigma}{4}z^3 - \frac{2-\sigma}{8}(x^2+y^2)z - \tfrac{3}{4}h^2 z + \tfrac{1}{2}h^3\right].$$

Hence

$$\chi = -p_0 h^{-3} xy\left[\frac{3-\sigma}{80}z^5 - \frac{2-\sigma}{48}(x^2+y^2)z^3 - \tfrac{1}{8}h^2z^3 + \tfrac{1}{4}h^3z^2\right] + z\chi_1'' + \chi_0'' \quad \dots \quad (25)$$

χ_1'' and χ_0'' being functions of x and y only. It may be readily shown that χ_1'' and χ_0'' satisfy the following equations respectively :

$$\nabla_1{}^2\chi_1'' = -\frac{1-\sigma}{8}p_0 h^{-3} xy(x^2+y^2) - \tfrac{3}{4}p_0 h^{-1} xy \quad \dots \quad (26)$$

and $\nabla_1{}^2\chi_0'' = \tfrac{1}{2}p_0 xy.$ ... (27)

Equations (26) and (27) on integration yield the following values for χ_1'' and χ_0'' as particular solutions :

$$\chi_1'' = -\frac{1-\sigma}{28}\frac{p_0}{h^3}xy(x^2+y^2)^2 - \tfrac{1}{16}p_0 h^{-1} xy(x^2+y^2) \dots \quad (28)$$

$$\chi_0'' = \tfrac{1}{24}p_0 xy(x^2+y^2) \quad \dots \quad (29)$$

We might add to each of (28) and (29) any arbitrary plane harmonic function; but it is unnecessary to do so as they may be supposed included in the functions Θ_1 and Θ_0.

Thus, we may now write for the general value of χ, the expression:—

$$\chi = -\frac{p_0}{h^3}xy\left[\frac{3-\sigma}{80}z^5 - \frac{2-\sigma}{48}(x^2+y^2)z^3 - \tfrac{1}{8}h^2z^3 + \tfrac{1}{4}h^3z^2\right]$$

$$-\frac{p_0}{h^3}z\left[\frac{1-\sigma}{2^8}xy(x^2+y^2)^2 + \tfrac{1}{16}h^2xy(x^2+y^2)\right]$$

$$+\tfrac{1}{24}p_0xy(x^2+y^2). \qquad \ldots (30)$$

With the help of this we may now write down the expressions for X_x, Y_y, X_y as follows:

$$X_x = \frac{1}{4}\frac{p_0}{h^3}xyz(3h^2-2z^2) + \frac{1}{8}\frac{p_0}{h^3}xyz(x^2+y^2) + \frac{p_0}{h^3}\frac{2-\sigma}{8}xyz^3$$

$$-\frac{p_0}{h^3}xyz\left[\frac{1-\sigma}{2^6}(3x^2+5y^2) + \frac{3}{8}h^2\right] + \frac{1}{4}p_0xy \ldots (31)$$

$$Y_y = \frac{1}{4}\frac{p_0}{h^3}xyz(3h^2-2z^2) + \frac{3}{8}\frac{p_0}{h^3}xyz(x^2+y^2) + \frac{p_0}{h^3}\frac{2-\sigma}{8}xyz^3$$

$$-\frac{p_0}{h^3}xyz\left[\frac{1-\sigma}{2^6}(5x^2+3y^2) + \frac{3}{8}h^2\right] + \frac{1}{4}p_0xy \ldots (32)$$

and

$$X_y = \frac{p_0}{h^3}\left[\frac{3-\sigma}{80}z^5 - \frac{2-\sigma}{16}(x^2+y^2)z^3 - \tfrac{1}{8}h^2z^3 + \tfrac{1}{4}h^3z^2\right]$$

$$+\frac{p_0}{h^3}\left[\frac{1-\sigma}{2^8}(5x^4+18x^2y^2+5y^4) + \tfrac{3}{16}h^2(x^2+y^2)\right]z$$

$$-\tfrac{1}{8}p_0(x^2+y^2) \qquad \ldots (33)$$

It is easy to shew that the strain components are given by the following equations:

$$e_{xx} = \frac{\partial u}{\partial x} = \frac{1+\sigma}{E}p_0h^{-3}xy\left[\tfrac{1}{4}h^3 + \tfrac{3}{8}h^2z + \frac{1-\sigma}{2^6}z(5x^2+3y^2)\right.$$

$$\left.-\frac{2-\sigma}{8}z^3\right] \qquad \ldots (34)$$

$$e_{yy} = \frac{\partial v}{\partial y} = \frac{1+\sigma}{E} p_0 h^{-3} xy \left[\tfrac{1}{4} h^3 + \tfrac{3}{8} h^2 z + \frac{1-\sigma}{2^6} z (3x^2 + 5y^2) - \frac{2-\sigma}{8} z^3 \right] \quad \ldots \quad (35)$$

$$e_{zz} = \frac{\partial w}{\partial z} = \frac{1+\sigma}{E} p_0 h^{-3} xy \left[-\tfrac{1}{2} h^3 - \tfrac{3}{4} h^2 z - \frac{\sigma}{8} (x^2 + y^2) z + \frac{1+\sigma}{4} z^3 \right] \quad \ldots \quad (36)$$

$$e_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \frac{1+\sigma}{E} p_0 h^{-3} \left[\tfrac{1}{8} h^3 x (x^2 + 3y^2) - \tfrac{1}{8} xz^2 (x^2 + 3y^2) + \tfrac{1}{2} h^4 x - \tfrac{3}{4} h^2 xz^2 + \tfrac{1}{4} xz^4 \right] \quad \ldots \quad (37)$$

$$e_{zx} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = \frac{1+\sigma}{E} p_0 h^{-3} \left[\tfrac{1}{8} h^2 y (3x^2 + y^2) - \tfrac{1}{8} yz^2 (3x^2 + y^2) + \tfrac{1}{2} h^4 y - \tfrac{3}{4} h^2 yz^2 + \tfrac{1}{4} yz^4 \right] \quad \ldots \quad (38)$$

$$e_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = \frac{1+\sigma}{E} p_0 h^{-3} \left[\frac{3-\sigma}{40} z^5 - \frac{2-\sigma}{8} (x^2 + y^2) z^3 - \frac{1}{4} h^2 z^3 + \frac{1}{2} h^3 z^2 + \frac{1-\sigma}{2^7} (5x^4 + 18x^2y^2 + 5y^4) z + \frac{3}{8} h^2 (x^2 + y^2) z - \tfrac{1}{4} h^3 (x^2 + y^2) \right] \quad \ldots \quad (39)$$

Solving the equations (34), (35), (36) we obtain the following expressions for the displacement components at any point within the body:

$$u = \frac{1+\sigma}{E} p_0 h^{-3} x^2 y \left[\frac{1}{8} h^3 + \frac{3}{16} h^2 z + \frac{1-\sigma}{2^8} z (5x^2 + 6y^2) - \frac{2-\sigma}{16} z^3 \right] + f_1 (y, z) \quad \ldots \quad (40)$$

$$v = \frac{1+\sigma}{E} p_0 h^{-3} xy^2 \left[\frac{1}{8} h^3 + \frac{3}{16} h^2 z + \frac{1-\sigma}{2^8} z (6x^2 + 5y^2) - \frac{2-\sigma}{16} z^3 \right] + f_2 (z, x) \quad \ldots \quad (41)$$

$$w = \frac{1+\sigma}{E} p_0 h^{-3} xy \left[-\frac{1}{2} h^3 z - \frac{3}{8} h^2 z^2 - \frac{\sigma}{16} (x^2 + y^2) z^2 + \frac{1+\sigma}{16} z^4 \right] + f_3 (x, y) \quad \ldots \quad (42)$$

where f_1, f_2, f_3 are three arbitrary functions of y, z; z, x and x, y respectively.

Substituting these values of u, v, w in equations (37), (38), and (39) we obtain

$$\frac{\partial f_3}{\partial y} + \frac{\partial f_2}{\partial z} = \frac{1+\sigma}{E} p_0 h^{-3} \Big[\frac{1}{2} h^3 xz - \frac{3}{8} h^2 xz^2 + \frac{\sigma}{16} x^3 z^2 + \frac{3-\sigma}{16} xz^4 - \tfrac{1}{8} x^3 z^2 + \tfrac{3}{16} xy^2 h^2 - \frac{1-\sigma}{2^8} xy^2 (6x^2 + 5y^2) + \tfrac{1}{8} h^2 x^3 + \tfrac{1}{2} h^4 x \Big] \quad \ldots \quad (43)$$

$$\frac{\partial f_1}{\partial z} + \frac{\partial f_3}{\partial x} = \frac{1+\sigma}{E} p_0 h^{-3} \Big[\frac{1}{2} h^3 yz - \frac{3}{8} h^2 yz^2 + \frac{\sigma}{16} y^3 z^2 + \frac{3-\sigma}{16} yz^4 - \tfrac{1}{8} y^3 z^2 + \tfrac{3}{16} x^2 y h^2 - \frac{1-\sigma}{2^8} x^2 y (5x^2 + 6y^2) + \tfrac{1}{8} h^2 y^3 + \tfrac{1}{2} h^4 y \Big] \quad \ldots \quad (44)$$

$$\frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial y} = \frac{1+\sigma}{E} p_0 h^{-3} \Big[\frac{3-\sigma}{40} z^5 - \frac{2-\sigma}{16} (x^2 + y^2) z^3 - \tfrac{3}{8} h^3 (x^2 + y^2) + \tfrac{3}{16} h^2 z (x^2 + y^2) - \tfrac{1}{4} h^2 z^3 + \tfrac{1}{2} h^3 z^2 + \frac{1-\sigma}{2^8} z (5x^4 + 5y^4) \Big] \quad \ldots \quad (45)$$

Equations (43), (44) and (45) are satisfied by assuming

$$f_1 = \frac{1+\sigma}{E} p_0 h^{-3} \Big[\frac{3-\sigma}{80} yz^5 - \frac{2-\sigma}{48} y^3 z^3 - \frac{1}{8} h^3 y^3 + \frac{1}{16} h^2 y^3 z - \frac{1}{8} h^2 yz^3 + \frac{1-\sigma}{2^8} y^5 z + \frac{1}{4} h^3 yz^2 \Big] \quad \ldots \quad (46)$$

$$f_2 = \frac{1+\sigma}{E} p_0 h^{-3} \Big[\frac{3-\sigma}{80} xz^5 - \frac{2-\sigma}{48} x^3 z^3 - \frac{1}{8} h^3 x^3 + \frac{1}{16} h^2 x^3 z - \frac{1}{8} h^2 xz^3 + \frac{1-\sigma}{2^8} x^5 z + \frac{1}{4} h^3 xz^2 \Big] \quad \ldots \quad (47)$$

$$f_3 = \frac{1+\sigma}{E} p_0 h^{-3} xy \Big[\frac{1}{2} h^4 + \frac{1}{16} h^2 (x^2 + y^2) - \frac{1-\sigma}{2^8} (x^2 + y^2)^2 \Big] \quad \ldots \quad (48)$$

The displacement components may now be written as

$$u = \frac{1+\sigma}{E} p_0 h^{-3} y \left[\frac{1}{8} h^3 x^2 + \frac{3}{16} h^2 x^2 z + \frac{1-\sigma}{2^8} x^2 z \, (5x^2 + 6y^2) \right.$$

$$- \frac{2-\sigma}{16} x^2 z^3 + \frac{3-\sigma}{80} z^5 - \frac{2-\sigma}{40} y^2 z^3 - \frac{1}{8} h^3 y^2$$

$$\left. + \frac{1}{16} h^2 y^2 z - \frac{1}{8} h^2 z^3 + \frac{1-\sigma}{2^8} y^4 z + \frac{1}{4} h^3 z^2 \right] \quad \ldots \quad (49)$$

$$v = \frac{1+\sigma}{E} p_0 h^{-3} x \left[\frac{1}{8} h^3 y^2 + \frac{3}{16} h^2 y^2 z + \frac{1-\sigma}{2^8} y^2 z \, (6x^2 + 5y^2) \right.$$

$$- \frac{2-\sigma}{16} y^2 z^3 + \frac{3-\sigma}{80} z^5 - \frac{2-\sigma}{48} x^2 z^3 - \frac{1}{8} h^3 x^2$$

$$+ \frac{1}{16} h^2 x^2 z - \frac{1}{8} h^2 z^3 + \frac{1-\sigma}{2^8} x^4 z + \frac{1}{4} h^3 z^2 \quad \ldots \quad (50)$$

$$w = \frac{1+\sigma}{E} p_0 h^{-3} xy \left[-\frac{1}{2} h^3 z - \frac{3}{8} h^2 z^2 - \frac{\sigma}{16} (x^2 + y^2) z^2 + \frac{1+\sigma}{16} z^4 \right.$$

$$\left. + \frac{1}{2} h^4 + \frac{1}{16} h^2 (x^2 + y^2) - \frac{1-\sigma}{2^8} (x^2 + y^2)^2 \right] \quad \ldots \quad (51)$$

From the expressions (34), (35) and (39) it is easy to show that the middle plane is slightly stretched. The middle plane is given by $z = 0$, for which we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{1+\sigma}{4E} p_0 xy,$$

and
$$\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = -\frac{1+\sigma}{4E} p_0 (x^2 + y^2).$$

The stress resultants and stress couples* are given by the following equations:

$$T_1 = \tfrac{1}{2} p_0 h x y \, ; \quad T_2 = \tfrac{1}{2} p_0 h x y$$

$$S_1 = \tfrac{1}{6} p_0 h^3 - \tfrac{1}{4} p_0 h (x^2 + y^2)$$

$$S_2 = -\tfrac{1}{6} p_0 h^3 + \tfrac{1}{4} p_0 h (x^2 + y^2)$$

$$N_1 = \frac{p_0}{12} y (3x^2 + y^2) + \tfrac{3}{10} p_0 h^2 y$$

$$N_2 = \frac{p_0}{12} x (x^2 + 3y^2) + \tfrac{3}{10} p_0 h^2 x.$$

* These notations too have been borrowed from Love.

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NOTES AND QUESTIONS.

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Notes and Questions.

On a formula for Geodesic Curvature.

1. Reference may be made to Darboux's *Leçons Sur la theorie general des Surfaces* Part III, pp 121 *et seq*, for various definitions of the geodesic curvature of a curve on a surface. Of these, two are usually adopted. One defines the geodesic curvature at any point A of a curve on a surface, as the curvature of the plane curve formed by the orthogonal projection of the given curve on the tangent plane to the surface at A. A second definition, which bears an analogy to the definition of curvature in plane geometry, is due to Liouville, *viz.*:

$$\text{geodesic curvature} = \text{Lt.} \frac{\text{angle between consecutive geodesic tangents}}{\text{element of arc}}.$$

Starting from the first definition, the formula

$$\frac{1}{\gamma} = \frac{\sin\omega}{\rho} \quad \dots \quad (1)$$

due to Liouville, is easily obtained with the help of Meusnier's theorem. Here ρ and γ denote the radii of circular and geodesic curvatures at any point on the skew curve, while ω denotes the angle between the principal normal and the normal to the surface at the point. The first definition is proved to be equivalent to the second by using a theorem of Gauss.*

2. We are concerned in this note with the method of deriving the formula (1) starting from the second definition. The method usually adopted, due to Liouville,† may be found in text-books.‡ The following is a different method, starting from first principles.

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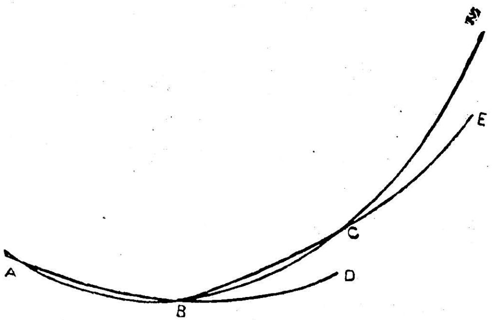
* Eisenhart : *Differential Geometry*, Articles 57 and 89.
Darboux : *Theorie des surfaces*, Part II, Art. 490 ; Part III, Arts. 641-642.

† Liouville's edition of Monge's *Applications de l'Analyse a la Geometrie*. Appendix, p. 576

‡ Forsyth : *Differential Geometry*, p 151 (Art. 104).
Salmon and Rogers : *Analytical Geometry of three dimensions*, Vol. I, p. 422.
Bell : *Co-ordinate Geometry of three dimensions*, p. 371.



Let A, B, C be three neighbouring points on the curve Σ on a surface such that arc AB = arc BC = ds, and let ABD and BCE be geodesics joining AB and BC.

Let (a, a', a'') denote the direction cosines of the common tangent at A, *i. e.*, of the limiting position of the straight line AB. Let 'd' and 'δ' be used to denote differentiations along the curve Σ, and along the geodesic tangent at A respectively, so that the direction cosines of the consecutive tangent to Σ, *i. e.*, of BC in the limit are $(a + da, a' + da', a'' + da'')$, and the direction cosines of the consecutive tangent to the geodesic, *i. e.*, of BD in the limit are $(a + \delta a, a' + \delta a', a'' + \delta a'')$. Then we have,

$$\left.\begin{aligned} \frac{1}{\rho^2} &= \left(\frac{da}{ds}\right)^2 + \left(\frac{da'}{ds}\right)^2 + \left(\frac{da''}{ds}\right)^2 \\ \frac{1}{\rho_0^2} &= \left(\frac{\delta a}{ds}\right)^2 + \left(\frac{\delta a'}{ds}\right)^2 + \left(\frac{\delta a''}{ds}\right)^2 \end{aligned}\right\} \quad \dots \quad (2)$$

where ρ and ρ_0 denote the radii of circular curvature of the curve and of the geodesic at A.

Now from Liouville's definition, $\frac{1}{\gamma} = \text{Lt.} \frac{C\hat{B}D}{ds}$. If $\delta\theta$ denotes the angle between two lines whose direction cosines are (l, m, n) and

$(l+\delta l,\ m+\delta m,\ n+\delta n)$, it can be shown that*

$$\delta\theta^2 = (\delta l^2 + \delta m^2 + \delta n^2) + O(\delta\theta^4).$$

Therefore $$\frac{1}{\gamma^2} = \text{Lt.}\ \frac{(da-\delta a)^2 + (da'-\delta a')^2 + (da''-\delta a'')^2}{ds^2}$$

$$= \frac{1}{\rho^2} + \frac{1}{\rho_0^2} - 2\Sigma\left(\frac{da}{ds}\,\frac{\delta a}{ds}\right) \quad \text{from (2).}$$

If (b, b', b''), (β, β', β'') denote the direction cosines of the principal normals to Σ and to its geodesic tangent at A, we have by the Serret-Frenet formulæ

$$\frac{da}{ds} = \frac{b}{\rho};\ \frac{\delta a}{\delta s} = \frac{\beta}{\rho_0};\ \text{etc.}$$

Therefore $$\frac{1}{\gamma^2} = \frac{1}{\rho^2} + \frac{1}{\rho_0^2} - \frac{2\cos\omega}{\rho\rho_0} \quad \dots \quad (3)$$

Now since ABD is a geodesic, its osculating plane at A contains the normal to the surface at A. Therefore $\frac{1}{\rho_0}$, the circular curvature of the geodesic, is also the curvature of the normal section ABD.

Hence, by Meusnier's theorem, $$\frac{1}{\rho_0} = \frac{1}{\rho}\cos\omega \quad \dots \quad (4)$$

Eliminating ρ_0, from (3) and (4) we obtain

$$\frac{1}{\gamma} = \frac{1}{\rho}\sin\omega.$$

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* Cf. Bell, *loc. cit.* p 23, example 10.

195330

On Identities connecting Inverse Trigonometrical Functions.

An inverse trigonometrical function may be defined as having its principal value, which is generally taken to be between $+\frac{\pi}{2}$ and $-\frac{\pi}{2}$ for the inverse sine, and between 0 and π for the inverse cosine. This gives rise to difficulties;

e.g.
$$\sin^{-1}(-\tfrac{1}{2}) \neq \cos^{-1}\left(\pm \frac{\sqrt{3}}{2}\right);$$

$$\tan^{-1} 2 + \tan^{-1} 3 = \tan^{-1}(-1) + \pi,$$

so that
$$\tan^{-1} a + \tan^{-1} b = \tan^{-1}\frac{a+b}{1-ab} + n\pi;$$

where n may have values 0, + 1 or − 1.

Moreover, it is convenient to take an inverse trigonometrical function as multiple-valued. With this definition we have to distinguish between two kinds of identities, both of which are different from ordinary identities. In $\tan^{-1} x = \cot^{-1} 1/x$, any value of the right-hand side equals some value of the left-hand side, and *vice versa*. In identities of this type, we may say that the domain of multiple values on the left side is co-extensive with the domain of multiple values on the right side.

The more usual type of identity is, however, of the second kind namely,

$$\sin^{-1} a = \cos^{-1}(\pm\sqrt{1-a^2}); \quad 2\tan^{-1} x = \tan^{-1}\frac{2x}{1-x^2}; \text{ etc.}$$

In these, any value on the left side is equal to some value on the right, but every value of the right side is not necessarily a value of the left side. The domain of values on the left side is in fact, a part of the domain of values on the right side, so that if $a = b$ then $b \neq a$ in the case of such identities. But if $a = b$ and $b = c$ in the above sense, then $a = c$, because domain of a is part of domain of b, and domain of b is part of domain of c. So the domain of a is part of domain of c, and so $a = c$ in the above sense.

A whole chapter is devoted in current trigonometrical text-books to identities connecting inverse trigonometrical functions, but mistakes are committed by not perceiving the above limitations on the identities involved.

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Certain Invariant Limit Theorems connected with a given segment of a Straight Line in Non-Euclidean Geometry.

1. *Preliminary.*

Let A_1ABB_1 be an isosceles bi-rectangular quadrilateral with right angles at A and B and with AA_1 and BB_1 as equal sides. We denote by $\phi\,(AA_1)$ the limit of the ratio $A_1B_1 : AB$ when B tends to A as limit, AA_1 being kept constant.

We require the following two Lemmas:

Lemma (1) If A_1ABB_1 be a tri-rectangular quadrilateral in which the angles A_1, A, B are right angles and if AA_1 being kept fixed, B tends to A, then the limit of the ratio $A_1B_1 : AB = \phi\,(AA_1)$.

Lemma (2). If B and C tend towards A as limit along fixed straight lines AX, AY and if the limit of the angle ABC equals a right angle, then the limit of the ratio of any two sides of this triangle is the same as if the angle ABC were exactly a right angle.

Lemma (2) follows from the well-known theorem that if in a triangle one angle becomes infinitesimal and neither of the other angles approaches a straight angle as a limit, then the side opposite the infinitesimal angle becomes infinitesimal as compared with either of the other sides. ... (2′)

We may also add to the theorem in (2′) that the limit of the ratio of the other two sides equals unity. ... (2″)

2. In the following theorems, AA_1 is a segment of a straight line; AX, A_1X_1 are perpendiculars to AA_1; B is a point in AX tending

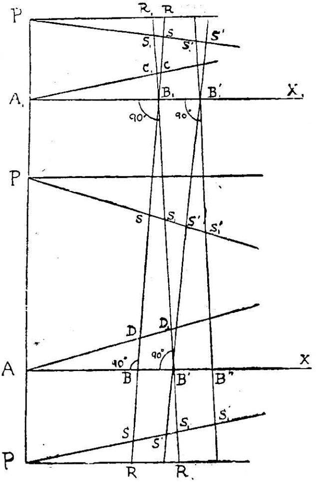

Figure for Hyperbolic Geometry.
A similar figure may be drawn for theorems in Elliptic Geometry.

to A as limit; B', B'' are points of AX and B_1, B_1', are points of A_1X_1 such that B_1B and B_1' B' are each perpendicular to AX, while B_1B' and $B_1'B''$ are each perpendicular to A_1X_1. P is a given point on AA_1 or AA_1 produced. Any line through P cuts BB_1, B_1B', $B'B_1'$, $B_1'B''$ in S, S_1, S', S_1', respectively.

THEOREM I. $$\underset{B \to A}{\text{Lt}} \frac{PS_1}{PS} = \frac{\phi(A_1P)}{\phi(AP)} \phi(AA_1)$$

For let the perpendicular to AA_1 through P cut BB_1, B_1B' in R, R_1 respectively.

Then $$\frac{PS_1}{PS} = \frac{PS_1}{PR_1} \times \frac{PR_1}{A_1B_1} \times \frac{A_1B_1}{AB} \times \frac{AB}{PR} \times \frac{PR}{PS} \qquad \dots \quad (1)$$

By Lemma (2), $\text{Lt}\frac{PS_1}{PR_1} = \sec S\hat{P}R$; and $\text{Lt}\frac{PR}{PS} = \cos S\hat{P}R$.

By Lemma (1), $\text{Lt}\frac{PR_1}{A_1B_1} = \phi(A_1P)$; $\text{Lt}\frac{A_1B_1}{AB} = \phi(AA_1)$

and $$\text{Lt}\frac{AB}{PR} = \frac{1}{\phi(AP)}$$

Substituting these values in (1) the theorem follows.

THEOREM II. $$\underset{B \to A}{\text{Lt}} \left[\frac{PS.PS'}{PS_1^2}\right] = \left[\frac{\phi(AP)}{\phi(A_1P)}\right]^2$$

By Theorem I, $\text{Lt}\frac{PS_1}{PS} = \frac{\phi(A_1P)}{\phi(AP)} \phi(AA_1)$.

and $$\text{Lt}\frac{PS'}{PS_1} = \frac{\phi(AP)}{\phi(A_1P)} \phi(AA_1)$$

$\therefore$ by division $$\text{Lt}\left[\frac{PS.PS'}{PS_1^2}\right] = \left[\frac{\phi(AP)}{\phi(A_1P)}\right]^2.$$

It should be noted that Theorems I and II are independent of the direction of PS, but depend on the position of P on AA_1.

THEOREM III. $$\underset{B \to A}{\text{Lt}} \frac{PS'}{PS} = [\phi(AA_1)]^2.$$

The theorem follows by multiplying the two limits put down in the proof of Theorem II.

THEOREM IV. $\underset{B \to A}{\text{Lt}} \left[\dfrac{PS \,.\, PS_1'}{PS' \,.\, PS_1}\right] = 1.$

By Theorem I, Lt $\dfrac{PS'}{PS}$ is the same as Lt $\dfrac{PS_1'}{PS'}$. Hence the theorem follows.

Theorem III shows that Lt PS' / PS is independent of the position of P on AA_1, as well as the direction of PS, while from Theorem IV Lt $PS \,.\, PS_1' / PS' \,.\, PS_1$ is an absolute constant.

3. *Special Cases of Theorems I and II.*

Case (i) *When P is at A_1.*

Let any line through A_1 cut BB_1, B_1B' in C, C_1 respectively.

Then by Theorem I,

$$\text{Lt}\,\frac{A_1C_1}{A_1C} = \frac{\phi\,(0)}{\phi\,(AA_1)} \,.\, \phi\,(AA_1) = 1.$$

$\phi\,(0)$ being taken as 1.

Case (ii) *When P is the mid-point of AA_1.*

Here $\phi\,(A_1P) = \phi\,(AP).$

$\therefore$ By Theorem I, Lt $\dfrac{PS_1}{PS} = \phi\,(AA_1).$

By Theorem II, Lt $\left[\dfrac{PS \,.\, PS'}{PS_1^2}\right] = 1.$

Case (iii) *When P is at A.*

Let any line through A cut BB_1, B_1B' in D, D_1 respectively.

By Theorem I, Lt $\dfrac{AD_1}{AD} = \dfrac{\phi\,(AA_1)}{\phi\,(0)} \,.\, \phi\,(AA_1) = [\,\phi\,(AA_1)]^2$

The results in cases (i) and (iii) may be proved directly in the same manner as the general theorem, and then it will not be necessary to assume that $\phi\,(0) = 1$.

4. *An application.*

To prove that $\phi\,(x + y) + \phi\,(x - y) = 2\,\phi\,(x) \,.\, \phi\,(y)$.

Let AA_1 be a segment of length x and let A_2A_3 be the segment of the line AA_1 whose length is $2y$ whose mid-point is A_1. Draw lines AX, A_1X_1, A_2X_2, A_3X_3 perpendicular to AA_1. Let B be a point of AX which tends to A as limit. Let the perpendicular to AX through B cut A_1X_1, A_2X_2, A_3X_3 in B_1, B_2, B_3 and let the perpendicular to A_1B_1 from B_1 cut A_2B_2, A_3B_3 in B'_2, B'_3 respectively.

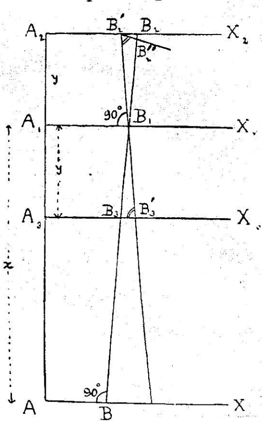

Figure for Hyperbolic Geometry. A similar figure may be drawn to illustrate applications to Elliptic Geometry

Then $A_2\,B_2'\;B_3'\;A_3$ is an isosceles bi-rectangular quadrilateral and $A_2\,B'_2 = A_3\,B'_3$ and $B_1\,B'_2 = B_1\,B'_3$.

Also, it is easy to see that in both geometries $\angle B_1B'_2B_2$ is obtuse and $\angle B_1B'_3B_3$ is acute so that B''_2 exists in B_1B_2 so that $\angle B_1B'_2B''_2 = \angle B_1B'_3B_3$.

Then the triangle $B_2B'_2B''_2$ has the angle at B'_2 infinitesimal while the other angles tend to right angles as limits.

So by result (8″), $\text{Lt}\dfrac{B_2\,B'_2}{B'_2\,B''_2} = \text{Lt}\dfrac{B_2B'_2}{B_3B'_3} = 1.$

But $\text{Lt}\dfrac{B_2B'_2}{A_2B'_2} = \text{Lt}\left[\dfrac{A_2B_2}{A_2B'_2} - 1\right] = \dfrac{\phi\,(x+y)}{\phi\,(x)\;\phi\,(y)} - 1$ by Theorem I.

and $\text{Lt}\dfrac{B_3\,B'_3}{A_3B'_3} = \text{Lt}\left[1 - \dfrac{A_3B_3}{A_3B'_3}\right] = 1 - \dfrac{\phi\,(x-y)}{\phi\,(x)\;\phi\,(y)}$,,

∴ By division, it follows that

$$\phi\,(x+y) - \phi\,(x)\;\phi\,(y) = \phi\,(x)\;\phi\,(y) - \phi\,(x-y)$$

or
$$\phi\,(x+y) + \phi\,(x-y) = 2\;\phi\,(x)\;\phi\,(y).$$

The particular case $\phi\,(2x) = 2\,[\phi\,(x)]^2 - 1$ can be independently proved by taking A at A_3 and following exactly the same method.

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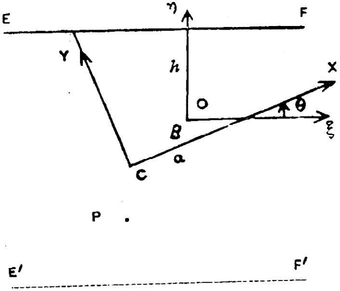
Solutions.

Question 150.

(M. T. NARANIENGAR, M.A.):—If a rectangular lamina always immersed in a fluid rotate (in its own plane) about a fixed point in the lamina, find the locus of the centre of pressure (i) in the lamina, (ii) in space.

Solution by M. L. Chandratreya.

The method of investigation of the loci is the same whatever be the shape of the lamina. The first part of the problem, *viz.*, the locus of the centre of pressure in the lamina is dealt with in text-books on hydrostatics [*vide*, for example, Besant and Ramsay's *Hydrostatics*, (7th edition), pp. 44-45, where it is shown that the locus is an ellipse.]



In the figure, O is the fixed point about which the lamina turns, C its centre of inertia, CX, CY the principal axes of the momental ellipse at C. EF is the line in which the lamina intersects the free surface of the liquid. $O\xi$, $O\eta$ are axes fixed in space, being parallel to and perpendicular to EF. Let (α, β) denote the position of O relative to CX, CY. P is the centre of pressure, having co-ordinates (x_1, y_1) relative to CX, CY and (ξ, η) relative to $O\xi$, $O\eta$. E′F′ is the line parallel to EF and as far below C as EF is above C. Let a, b denote the radii of

gyration of the lamina about CX and CY; let h be the distance of O from EF and θ the angle that CX, CY make with $O\xi$, $O\eta$.

It is well-known that P is the pole of E′F′ with respect to the momental ellipse at C.*

(i) Now the equation to EF referred to CX, CY is

$$x \sin\theta + y\cos\theta = h + \alpha\sin\theta + \beta\cos\theta.$$

Hence the equation to E′F′ is

$$x\sin\theta + y\cos\theta = -(h + \alpha\sin\theta + \beta\cos\theta).$$

Since this is the polar of P (x_1, y_1) with respect to the momental ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

we get

$$x_1 = \frac{-a^2\sin\theta}{h + \alpha\sin\theta + \beta\cos\theta},\quad y_1 = \frac{-b^2\cos\theta}{h + \alpha\sin\theta + \beta\cos\theta} \quad \ldots \quad (1)$$

Eliminating θ between the equations (1), we get the locus of P in the lamina to be the ellipse

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{h^2}\left(\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + 1\right)^2.$$

(ii) The relations connecting (ξ, η) and (x_1, y_1) are

$$\left.\begin{aligned}\xi &= (x_1 - \alpha)\cos\theta - (y_1 - \beta)\sin\theta\\ \eta &= (x_1 - \alpha)\sin\theta + (y_1 - \beta)\cos\theta\end{aligned}\right\} \quad \ldots \quad (2)$$

Substituting the values of x_1, y_1 from (1) and expressing $\sin\theta$, $\cos\theta$ in terms of $t = \tan\frac{1}{2}\theta$, we see that ξ, η are rational quartics in t. Hence the locus of P in space is a unicursal quartic curve.

Question 1076.

(S. RAMANUJAN):—Prove that

(i) $(7.20^{\frac{1}{3}} - 19)^{\frac{1}{6}} = \left(\frac{5}{3}\right)^{\frac{1}{3}} - \left(\frac{2}{3}\right)^{\frac{1}{3}}$

(ii) $\left\{4\left(\frac{2}{3}\right)^{\frac{1}{3}} - 5\left(\frac{1}{3}\right)^{\frac{1}{3}}\right\}^{\frac{1}{2}} = \left(\frac{4}{9}\right)^{\frac{1}{3}} - \left(\frac{2}{9}\right)^{\frac{1}{3}} + \left(\frac{1}{9}\right)^{\frac{1}{3}}$.

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* Besant and Ramsay, *loc. cit*, p. 40.

Solution by R. Kothandaraman.*

The exponents $\frac{1}{8}$ and $\frac{1}{6}$ should be interchanged.

(i) Put $x^3 = \frac{5}{3}$ and $y^3 = \frac{2}{3}$; then

$$(x-y)^6 = x^6 - 6x^5y + 15x^4y^2 - 20x^3y^3 + 15x^2y^4 - 6xy^5 + y^6$$

$$= \frac{25 - 20.5.2 + 4}{9} - \frac{6.5 - 15.2}{3}x^2y + \frac{15.5 - 6.2}{3}xy^2$$

$$= -19 + 7\,(20)^{\frac{1}{3}}.$$

Hence $$\left\{7.20^{\frac{1}{3}} - 19\right\}^{\frac{1}{6}} = \left(\frac{5}{3}\right)^{\frac{1}{3}} - \left(\frac{2}{3}\right)^{\frac{1}{3}}.$$

(ii) Setting $t^3 = 2$,

$$\left(\frac{t^2 - t + 1}{\sqrt[3]{9}}\right)^8 = \frac{(t^3+1)^8}{9^{\frac{8}{3}}(t+1)^8} = \frac{3^8}{9^{\frac{8}{3}}.9\,(16t^2 + 20t + 25)}$$

after expansion and reduction,

$$= \frac{3^{\frac{2}{3}}(4t-5)}{(4t)^3 - (5)^3}$$

$$= \frac{4t-5}{3^{\frac{1}{3}}}.$$

Hence $$\left\{4\left(\frac{2}{3}\right)^{\frac{1}{3}} - 5\left(\frac{1}{3}\right)^{\frac{1}{3}}\right\}^{\frac{1}{8}} = \left(\frac{4}{9}\right)^{\frac{1}{3}} - \left(\frac{2}{9}\right)^{\frac{1}{3}} + \left(\frac{1}{9}\right)^{\frac{1}{3}}.$$

Question 1384.

(MARTYN THOMAS) :—A point P revolves in a circle of radius a with angular velocity ω, while a straight line PQ rotates in the opposite direction with angular velocity ω'. Show that the envelope of PQ is a hypo-cycloid the radii of whose fixed and moving circles are respectively

$$\left(1 + \frac{\omega}{\omega'}\right) a \quad \text{and} \quad \frac{\omega'}{2\omega} a.$$

Deduce the corresponding result when the straight line PQ rotates in the same direction as P.

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* *Vide J. I. M. S.*, Vol. XVII, pp. *154—156* for another solution.

Solution by G. V. Krishnaswami.*

There is a slight mistake in the question. The radius of the moving circle ought to be $\frac{\omega}{2\omega'}\, a$.

Taking the centre O as origin and the line OPQ in its initial position as the x-axis, the equation to the line PQ after a time t is

$$x \sin \omega' t + y \cos \omega' t = a \sin (\omega + \omega')t.$$

From page 348, Art. 421, Edwards' *Differential Calculus* the envelope of this line is $r^2 = n^2 a^2 + (1 - n^2)\, p^2$

where $n = \frac{\omega + \omega'}{\omega'} = 1 + \frac{\omega}{\omega'}.$

Further the equation of an epi-cycloid formed by the rolling of a circle of radius B on a circle of radius A is

$$r^2 = A^2 + 4\,(A + B)\,B \left(\frac{p}{A + 2B} \right)^2$$

Ibid page, 345, Art. 411.

Comparing the two equations, we have

$$A = na,\quad B = \frac{a}{2}(1 - n) = -\frac{\omega}{2\omega'}\, a.$$

Hence the result.

When PQ rotates in the same direction as P the result is got by substituting $-\omega'$ for ω'. Hence the envelope is an epicycloid the radii of whose fixed and moving circles are $\left(1 - \frac{\omega}{\omega'}\right) a$ and $\frac{\omega}{2\omega'}\, a$, respectively.

Also solved by S. Audinarayanan, T. S. Venkataraman, M. V. Seshadri and B. Ramamurthi.

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* *Vide* also *J. I. M. S.*, Vol. XVIII (2), p. 32, for a geometrical proof of a similar problem.

Question 1390.

(S. D. CHOWLA) :—Prove that

$$\sum_{n=1}^{\infty}\left[\frac{e^{-n^2\pi}}{(1-e^{-2\pi})(1-e^{-4\pi})\ldots(1-e^{-2\pi n})}\right]^2=\frac{\sqrt{2}\,\Gamma(\frac{3}{4})\,e^{-\frac{\pi}{12}}-\pi^{\frac{1}{4}}}{\pi^{\frac{1}{4}}}$$

Solution by the Proposer.

Assuming that

$$\frac{1}{(1-qz)(1-q^2z)(1-q^3z)\ldots}=1+\sum_{n=1}^{\infty}\frac{B_n z^n}{(1-qz)(1-q^2z)\ldots(1-q^nz)},$$

it is easily proved that

$$B_n=\frac{q^{n^2}}{(1-q)(1-q^2)\ldots(1-q^n)}.$$

Let $z=1$, and write q^2 for q; and we get

$$\frac{1}{(1-q^2)(1-q^4)(1-q^6)\ldots}=1+\sum_{r=1}^{\infty}\left[\frac{q^{r^2}}{(1-q^2)(1-q^4)\ldots(1-q^{2r})}\right]^2$$

But it is well-known that

$$(1-q^2)(1-q^4)(1-q^6)\ldots=K^{\frac{1}{2}}\pi^{-\frac{1}{2}}q^{-\frac{1}{12}}(2kk')^{\frac{1}{6}}.$$

Hence the result becomes

$$1+\sum_{r=1}^{\infty}\left[\frac{q^{r^2}}{(1-q^2)(1-q^4)\ldots(1-q^{2r})}\right]^2=K^{-\frac{1}{2}}\pi^{\frac{1}{2}}q^{\frac{1}{12}}(2kk')^{-\frac{1}{6}}.$$

Put $k'=k=\frac{1}{\sqrt{2}}$, so that $q=e^{-\pi}$ and

$$K=\int_0^{\frac{\pi}{2}}\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}}=\frac{\Gamma^2(\frac{1}{4})}{4\sqrt{\pi}}.$$

(Put $n=0$ in the result of Question 904, *J. I. M. S.*, Vol. X, pp. 368-369).

$$\therefore\ \frac{\sqrt{2}\,\Gamma(\frac{3}{4})\,e^{-\frac{\pi}{12}}-\pi^{\frac{1}{4}}}{\pi^{\frac{1}{4}}}=\sum_{n=1}^{\infty}\left[\frac{e^{-n^2\pi}}{(1-e^{-2\pi})(1-e^{-4\pi})\ldots(1-e^{-2n\pi})}\right]$$

Question 1396.

(V. RAMASWAMI AIYAR) :—Given four concyclic points (O_1, O_2, O_3, O_4), if the ortho-centres of the triangles whose vertices they are, be denoted by (H_1, H_2, H_3, H_4) in the appropriate manner, show that the set of points (H_1, H_2, H_3, H_4) as well as the six sets of points like (O_1, O_2, H_3, H_4) lie on circles equal to the circle ($O_1O_2O_3O_4$).

Solution by G. V. Krishnaswami, M. V. Seshadri, N. P. Subramanian, M. Venkataramayyar, S. Audinarayanan, and T. S. Venkataraman.

Let O_1H_3 and O_3H_1 meet the circle $O_1O_2O_3O_4$ again at P_3 and P_1 respectively. Then P_1 and P_3 are the images of H_1 and H_3 in O_2O_4. But O_2, P_3, O_4, P_1 lie on circle $O_1O_2O_3O_4$. Therefore O_2, H_3, O_4, H_1 lie on an equal circle. Similarly for the other five sets.

Again, if X, Y be the mid.-points of O_1O_3 and O_1O_4 respectively and if S be the centre of the circle $O_1O_2O_3O_4$ we have

$$O_2H_4 \parallel SX \text{ and } = 2\,SX;$$

$$O_2H_3 \parallel SY \text{ and } = 2\,SY.$$

Hence $$H_4H_3 \parallel XY \text{ and } = 2\,XY,$$

i.e., $H_4H_3 \parallel O_3O_4$ and equal to it. Similarly for the other lines H_3H_1, H_1H_2, etc. Hence H_1, H_2, H_3, H_4 lie on a circle equal to the circle $O_1O_2O_3O_4$.

Also solved analytically by G. V. Krishnaswami and N. Dorairajan.

Remarks by V. Ramaswami Aiyar.

In my question No. 1396, the real problem that I wanted to give has suffered an omission. I should have added 'On a sphere, can you find eight points lying four by four on eight equal circles in this manner ? What is the spherical type of eight such points ?'

Question 1398.

(MARTYN THOMAS):—Show that the singular solution of the differential equation

$$4\left(x+y\frac{dy}{dx}\right)^2+25\left(y^2-xy\frac{dy}{dx}\right)=0$$

is the envelope of a series of circles on the subnormal of an ellipse as diameter.

Solution by T. Totadri Iyengar, Kanwar Bahadur, G. V. Krishnaswami and M. V. Seshadri.

The p-discriminant equated to zero reduces to $y^2\,(9x^2-16y^2)=0$. Neither $y=0$, nor $y=\pm\frac{3}{4}x$ satisfies the equation. So the given equation has no singular solution, but $16y^2=9x^2$ represents the cuspidal locus.

Now, let P be the point $(a\cos\phi,\ b\sin\phi)$ on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

PN the ordinate and PG the normal at P. The subnormal is GN. The co-ordinates of G and N are $(e^2\,a\cos\phi,\ 0)$ and $(a\cos\phi,\ 0)$ respectively. A circle with GN as diameter is

$$y^2+(x-a\cos\phi)(x-ae^2\cos\phi)=0$$

or $$x^2+y^2-a\cos\phi\,.\,x\,(1+e^2)+a^2e^2\cos^2\phi=0.$$

The envelope of this family of circles is

$$x^2\,(1+e^2)^2=4e^2\,(x^2+y^2)$$

or $$x^2\,(1-e^2)^2=4e^2y^2$$

which is of the form $9x^2-16y^2=0$, if we put $e=\frac{1}{2}$.

Question 1402.

(S. D. CHOWLA):—Prove that

$$\Sigma\ \phi\,(n_1)\,.\,\lambda\,(n_2)=d\,(n)$$

where $d\,(n)$ denotes the sum of the divisors of n, $\phi\,(n)$ the number of numbers less than n and prime to it, $\lambda\,(n)$ the number of divisors of n; n_1 and n_2 are conjugate divisors of n, and the summation extends over all the divisors of n.

Example. d (15) = 24, and

$$\lambda(1)\,\phi(15)+\lambda(15)\,\phi(1)+\lambda(3)\,\phi(5)+\lambda(5)\,\phi(3)=24.$$

(1) *Solution and Remarks by the proposer.*

In order to prove a more general result, let

$$n = p^{\alpha}\,.\,q^{\beta}\,.\,r^{\gamma}\,\ldots,$$

$$\sigma^{a}(n)=\sum_{n/d} d^{a} \text{ (sum of } a\text{th powers of divisors } d \text{ of } n),$$

$$\phi_m(n)=n^m\left(1-\frac{1}{p^m}\right)\left(1-\frac{1}{q^m}\right)\left(1-\frac{1}{r^m}\right)\ldots$$

so that $\phi_1(n)$ = the $\phi(n)$ of this question.

Then following Bromwich: *Theory of Infinite Series*, pp. 494-495, it is easily proved that:

$$\frac{\zeta(s-m-a)}{\zeta(s-a)}=\sum_{1}^{\infty}\frac{n^a\,\phi_m(n)}{n_s},$$

Also we know that

$$\zeta(s)\,\zeta(s-a)=\sum_{1}^{\infty}\frac{\sigma_a(n)}{n^s},$$

whence

$$\zeta(s)\,\zeta(s-a)\,\frac{\zeta(s-m-a)}{\zeta(s-a)}=\sum_{1}^{\infty}\frac{\left[\sum_{n/d}\left\{d^a\,\phi_m(d)\,\sigma_a\left(\frac{n}{d}\right)\right\}\right]}{n^s}$$

$$=\zeta(s)\,\zeta(s-m-a)$$

$$=\sum_{1}^{\infty}\frac{\sigma_{a+m}(n)}{n^s}.$$

Hence we have

$$\sum_{n/d}\left\{d^a\,\phi_m(d)\,\sigma_a\left(\frac{n}{d}\right)\right\}=\sigma_{a+m}(n) \qquad \ldots \quad (1)$$

In particular, putting $a = 0$, $m = 1$, and allowing for the difference in notation, we get the required result.

Example.

Put $a = 0$, $n = 6$ in (1). Then the expression on the left is

$$4.1 + 2(2^m - 1) + 2(3^m - 1) + (2^m - 1)(3^m - 1)$$
$$= 1^m + 2^m + 3^m + 6^m$$
$$= \sigma_m(6) = \text{the expression on the right.}$$

(2) *Solution by T. Totadri Iyengar.*

If $1, r_1, r_2, r_3 \ldots \ldots \ldots, n$ are all the divisors of n, then $d(n) = 1 + r_1 + r_2 + \ldots \ldots + r_r + \ldots \ldots + n$. It is well-known that

$$\Sigma\, \phi(\alpha_m) = m,$$

where the Σ extends to all the quantities α_m which are divisors of m (1 and m inclusive).

Hence $d(n) = \Sigma\, \phi(\alpha_1) + \Sigma\, \phi(\alpha_{r_1}) + \ldots + \Sigma\, \phi(\alpha_{r_r})$
$$+ \ldots + \Sigma\, \phi(\alpha_n)$$

Considering a particular divisor n_1 of n, $\phi(n_1)$ must appear once and once only under the last Σ of the dexter. It must also, for similar reasons, appear under each of the expressions $\Sigma\, \phi(\alpha_{r_r})$ once only, whenever n_1 is a factor, *i.e.*, divisor of r_r.

Now n_1 is a divisor of n, $n_1 k_1$, $n_1 k_2$, ... $n_1 n_2$ where $1, k_1, k_2, \ldots \ldots n_2$ are all the divisors of n_2 the conjugate of n_1. Hence the number of times $\phi(n_1)$ appears on the right-side $= \lambda(n_2)$.

$$\therefore \quad d(n) = \Sigma\, \phi(n_1)\, \lambda(n_2).$$

Question 1410.

(HANSRAJ GUPTA):—If $\sum_{n=0}^{\infty} a_n x^n$ b ea convergent recurring series of the kth order, show that its sum is equal to $D_1 \div D_2$ where

$$D_1 \equiv \begin{vmatrix} 0, & s_1 x^{k-1}, & s_2 x^{k-2}, & \ldots & s_k \\ a_0, & a_1, & a_2, & \ldots & a_k \\ a_1, & a_2, & a_3, & \ldots & a_{k+1} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{k-1}, & a_k, & a_{k+1}, & \ldots & a_{2k-1} \end{vmatrix}$$

$$D_2 \equiv \begin{vmatrix} x^k, & x^{k-1}, & x^{k-2}, & \dots & x, & 1 \\ a_0, & a_1, & a_2, & \dots & a_{k-1}, & a_k \\ a_1, & a_2, & a_3, & \dots & a_k, & a_{k+1} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{k-1}, & a_k, & a_{k+1}, & \dots & a_{2k-2}, & a_{2k-1} \end{vmatrix}$$

and $$s_r = \sum_{n=0}^{r-1} a_n x^n.$$

Solution by G. V. Krishnaswami, T. Totadri Iyengar. B. Achyutarama Sastri, M. V. Seshadri, T. S. Venkataraman, S. Mahadevan, and K. Kuppuswamy Iyengar.

This problem is a generalisation of Question 1397 and can be proved on exactly the same lines.

If the scale of relation be $1+b_1x+b_2x^2+ \dots + b_k x^k$ the series is

$$\frac{N}{D} \equiv \frac{a_0+x(a_0b_1+a_1)+\dots+x^{k-1}(a_0 b_{k-1}+\dots+a_{k-1})}{1+b_1x+b_2x^2+\dots+b_k x^k}$$

$$= \frac{s_k+s_{k-1}x b_1+\dots+b_{k-2}s_2 x^{k-2}+b_{k-1}s_1 x^{k-1}}{1+b_1x+b_2x^2+\dots+b_k x^k}$$

and N, D and the b's satisfy the relations

$$\begin{array}{lllll} a_0b_k & +a_1 b_{k-1} & +a_2b_{k-2}+\dots+a_k & =0. & (1) \\ a_1b_k & +a_2b_{k-1} & +a_3b_{k-2}+\dots+a_{k+1} & =0. & (2) \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ a_{k-1}b_k & +a\, b_{k-1} & + \quad \dots+a_{2k-1} & =0. & (k) \\ x^k b_k & +x^{-1}b_{k-1} & + \quad \dots+1-D & =0. & (k+1) \\ 0.b_k & +s_1x^{k-1}b_{k-1} & + \quad \dots+s_k-N & =0. & (k+2) \end{array}$$

Expanding the determinant obtained from (1), (2), (3), ... (k) and $(k+1)$ we get

$$D = (-1)^k \frac{D_2}{\begin{vmatrix} a_0, & a_1, & a_2 & \cdots & a_{k-1} \\ a_1, & a_2, & a_3 & \cdots & a_k \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-1}, & a_k, & a_{k+1}, & \cdots & a_{2k-2} \end{vmatrix}}.$$

Similarly from (1), (2), (3), ... (k) and $(k+2)$ we get

$$N = (-1)^k \frac{D_1}{\begin{vmatrix} a_0, & a_1, & a_2, & \cdots & a_{k-1} \\ a_1, & a_2, & a_3, & \cdots & a_k \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{k-1}, & a_k, & a_{k+1}, & \cdots & a_{2k-2} \end{vmatrix}}.$$

Hence the sum of the recurring series is

$$\frac{N}{D} = \frac{D_1}{D_2}.$$

Question 1414.

(S. AUDINARAYANAN):—Evaluate

$$\int_0^\infty \frac{x^2}{\sinh^2 x}\, dx.$$

Solution by K. V. Vedantam.

$$I = \int_0 \left(\frac{x}{\sinh x}\right)^n dx = \int_0^\infty \left[\sum_{r=0}^{\infty} 2^n x^n e^{-nx} . e^{-2rx} . {}_{n+r-1}C_r \right] dx$$

$$= \sum_{r=0}^{\infty} 2^n\, n!\, \frac{{}_{n+r-1}C_r}{(2r+n)^{n+1}}.$$

Put $n = 2$, I becomes $\sum_{r=0}^{\infty} \frac{1}{(r+1)^2} = \frac{\pi^2}{6}$.

Also solved by S. D. Chowla, T. Totadri Iyengar, S Mahadevan, and Kanwar Bahadur.

Question 1427.

(S. D. CHOWLA):—Use Cauchy's theory of Reciprocal functions to prove Riemann's famous result

$$2^{1-s}\,\Gamma(s)\,\zeta(s)\cos\frac{s\pi}{2} = \pi^s \zeta(1-s).$$

Solution and Remarks by T. Totadri Iyengar and V. Thiruvenkatacharya.

Mr. K. B. Madhava has proved the above formula by just the required method, in his paper "On Riemann's Zeta Functions" in the *Journal of the Indian Mathematical Society*, Vol. IX (1917), pp. 141—148.

By Cauchy's theory of Reciprocal functions, it is known that if

$$\int_0^{\infty} \phi(x) \cos knx \, dx = \psi(n) \quad \ldots \quad (1)$$

then

$$\left.\begin{aligned}&\frac{\alpha}{2}\left[\tfrac{1}{2}\phi(0) + \phi(\alpha) + \phi(2\alpha) + \phi(3\alpha) + \ldots\ldots\right]\\ &= \tfrac{1}{2}\psi(0) + \psi(\beta) + \psi(2\beta) + \psi(3\beta) + \ldots\ldots\end{aligned}\right\} \quad \ldots \quad (2)$$

where

$$\alpha\beta = \frac{2\pi}{k}.$$

Again, it is also known that

$$\int_0^{\infty} t^{s-1}.\cos nt\, dt = \frac{\Gamma(s)}{n^s}\cos\frac{s\pi}{2} \quad \ldots \quad (3)$$

From (1), (2), (3), by writing $\phi(t) = t^{s-1}$, $\psi(n) = \frac{\Gamma(s)}{n^s}\cos\frac{s\pi}{2}$, we get the result of the question.

Questions for Solution.

Proposers of Questions are requested to send their own solutions along with their questions.

1542. (HEMRAJ):—

If $\prod_{k} (rp + k) = a + \sigma_r$ and $\prod_{r} (a + \sigma_r) = a^m + \sum_{\beta=1}^{\beta=m} (a^{m-\beta} S_\beta)$

where p is an odd prime, $a = \Pi(k)$; $n = p^{\mu-1} - 1$ (μ being a positive integer); $m = n$ or $n + 1$ according to circumstances; and S_β denotes the sum of the products (β at a time) of the numbers $\sigma_1, \sigma_2 \dots \sigma_m$; prove that

$$S_\beta \equiv 0 \bmod p^{\mu-1+8\beta}$$

or

$$\equiv 0 \bmod p^{\mu-1+\beta}$$

according as (i) k varies from 1 to $p-1$ and r from 1 to n,

or (ii) k runs through odd values from 1 to $p-2$ and r through even values from 2 to $2n$ or k through even values from 2 to $p-1$ and r through odd values from 1 to $2n+1$.

Deduce that the product of all the positive integers less than N and prime to it $\equiv (-1)$ mod. N where $N = p^\mu$ or $2p^\mu$.

1543. (C. N. SRINIVASIENGAR):—Construct examples of ruled surfaces possessing a singular generator, *i.e.*, one which meets all the other generators.

If the singular generator lies on the Hessian, show that it will be a double line on the surface.

1544. (S. NARAYANAN, B.A., L.T.):—

(i) If $p\alpha + q\beta + r\gamma = 0$ be an axis of a conic inscribed in the riangle of reference ABC, shew that its equation is

$$\sqrt{\alpha p(p^2 - q^2 - r^2 + 2qr\cos A)} + \sqrt{\beta q(q^2 - r^2 - p^2 + 2rp\cos B)} + \sqrt{\gamma r(r^2 - p^2 - q^2 + 2pq\cos C)} = 0.$$

(ii) If the join of the foci of a conic inscribed in a triangle ABC subtend angles θ, ϕ, ψ at the vertices A, B, C, shew that its equation is

$$\sin A \sqrt{\alpha/(\cos\theta - \cos A)} + \sin B \sqrt{\beta/(\cos\phi - \cos B)}$$
$$+ \sin C \sqrt{\gamma/(\cos\psi - \cos C)} = 0.$$

(iii) A conic being inscribed in the triangle ABC. if the joins of the vertices to a focus of the conic divide the angles A, B, C into parts α_1, α_2; β_1, β_2; γ_1, γ_2; show that its equation is

$$\sqrt{\alpha \sin A (\cot\alpha_1 + \cot\alpha_2)} + \sqrt{\beta \sin B (\cot\beta_1 + \cot\beta_2)}$$
$$+ \sqrt{\gamma \sin C (\cot\gamma_1 + \cot\gamma_2)} = 0.$$

1545, (K. J. SANJANA, M.A. AND G. R. JANI, B A.) :—Resolve the following numbers into factors :—

$$N_1 = 3^{52} + 2^{26},\ N_2 = 3^{27} + 2^{54} \text{ and } N_3 = 5^{25} - 2^{50}.$$

1546. (V. RAMASWAMI AIYAR, M.A.) :—Let ABC ... KL be a solid $(n+1)$-hedron, in space of n (or higher) dimensions, of uniform density and mass M.

Let a mass M_1 be made into $(n+1)$ equal particles, and one of these placed at each vertex of the figure; let a mass M_2 be made into ${}_{n+1}C_2$ equal particles, and one of these placed at the middle point of each edge, like AB, of the figure; let a mass M_3 be made into ${}_{n+1}C_3$ equal particles, and one of these placed at the centroid of each "3-edge," like ABC, of the figure; and so on till a mass M_n is made into ${}_{n+1}C_n$ equal particles, and one of these placed at the centroid of each "face" of the figure; then, let a mass M_{n+1} be placed at the centroid of the $(n+1)$-hedron.

Show that the system of particles so placed is equimomental with the solid simplex, provided that

(1) $M_1 + M_2 + M_3 + \dots\dots + M_n + M_{n+1} = M$;

and (2) $M_1 + \dfrac{M_2}{2} + \dfrac{M_3}{3} + \dots\dots + \dfrac{M_n}{n} + \dfrac{M_{n+1}}{n+1} = \dfrac{2M}{n+2}$.

[The work leading to this result, was suggested by Mr. A. Narasinga Rao's Note on Equimomental Systems in the last number of the Journal.]

Obituary.

H. BALAKRAM.

HOON BALAKRAM was born in Jallundher, Punjab, on the 21st April 1876. From his childhood he showed signs of remarkable intelligence, and was an exceptionally diligent student After qualifying himself for the Master of Arts degree in Physics of the Punjab University, he proceeded to England for further study and joined the St. John's College, Cambridge. In 1899 he passed the Mathematical Tripos and was placed fourth in the list of wranglers. The same year he was successful in the competitive examination for the Indian Civil Service.

Keenly interested in all intellectual pursuits, Balakram was naturally drawn towards the newly-started Indian Mathematical Club, of which he became one of the pioneer workers. He was twice elected President of the Society and was for several years on the Managing Committee. He was so much impressed by the need for improving the Society's library, that early in 1920 he made a generous donation of Rs. 1000 for the purpose, and organised a Committee for the purchase of back numbers of the leading mathematical journals. Balakram was a very genial figure at the Conferences of the Society, where he came into intimate contact with some of its promising members. In his Presidential Address at the Poona Conference he gave special encouragement to members of the Society who were engaged in throwing light on the History of Indian Mathematics. His keen intelligence, broad outlook, calmness and enthusiasm, and his very kindly and lovable character made him universally loved and respected.

In spite of his heavy official duties, Balakram devoted his leisure hours to mathematical study, where he had a very wide range of interest, the theory of relativity and the theory of numbers being his favourite subjects. On the latter he has published a number of interesting papers and notes in our Journal. Several ingenious questions and solutions by him appear under the title "H. Br." While he was on leave in Europe last year, he was working at a problem mentioned in Ramanujan's collected papers, namely, numbers which admit of being expressed as the sum of two cubes in two different ways, the number 1729 being the smallest of such numbers. Balakram devised a method for finding more of these, and had actually calculated all numbers of the kind below 20,000,000. The results of this investigation were read out at the last Conference of our Society at Nagpur. A month

before his death, when he was appointed a High Court Judge in Bombay, he was actively working in the Theory of Numbers, and has left behind him some interesting results.

By his sudden and untimely death the cause of Mathematics in India has lost a sincere worker, and all who knew him, a true and warm-hearted friend.

LIST OF PUBLISHED PAPERS BY H. BALAKRAM.

(The references are to the volumes and pages of this Journal).

1. *Common factors of* $\dfrac{n!}{m!\,(n-m)!}$: $(m=1, 2 \ldots n-1)$.
   Vol. I pp. 39—43.
2. *Curvature of curves defined by conjugate functions* :
   Vol I. pp. 218—222.
3. *Solid angles and potentials of plane discs* : Vol. II. pp. 3—8.
4. *Exact differential equations in quaternions* :
   Vol. II. pp. 127—133.
5. *Gauss' Measure of Curvature* : Vol. II. pp. 187—189.
6. *The general equation in the Algebra of Logic* :
   Vol. III. pp. 213—218.
7. *Mersenne's Numbers* : Vol. V. pp. 56—59.
8. *Presidential Address at the Third Conference of the Indian Mathematical Society, Lahore* : Vol. XIII. pp. 97—107.
9. *Presidential Address at the Fourth Conference of the Indian Mathematical Society, Poona* : Vol. XV, Supplement, pp.16—28.
10. *Some diophantine equations* : Vol. XVI. pp. 262—269.
11. *To find the factors of* N = 35821 : Vol. XVI. Part II, pp. *52—53*.
12. *Two well-known definite integrals* :
    Vol. XVI. Part II. pp. *117—118*.
13. *Indeterminate equations of the first degree* :
    Vol. XVI Part II pp. *179—181*.
14. *Solutions of congruences by means of divergent series* :
    Vol. XVII pp. 30—40.
15. *The integral part of a number, as a definite integral* :
    Vol. XVII Part II pp. *17—20*.
16. *Perpetual calender formulæ* : Vol XVII. Part II. pp. 145—149.

Questions for Solution. Nos. 1, 244, 245, 941, 942, 961, 964, 969.

Solutions to Questions. Nos. 1, 64, 142, 191, 276, 296, 301, 357 378, 508, 723, 800, 810, 846, 851, 874, 888, 943, 1109.

Reviews : Vol. II, pp. 149—151, and 223—228.

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during the months of April and May 1929.

1 Acta Mathematica, **52**, 3, 4.
2 American Journal of Mathematics, **51**, 2 (3 copies).
3 American Mathematical Monthly, **36**, 2 (3 copies). **36**, 3 (2 copies).
4 Annales of Mathematics, **30**, 1 (2 copies).
5 Astrophysical Journal, **69**, 1.
6 Bulletin of the Calcutta Mathematical Society, **21**, 1.
7 Bulletin des Science Mathematiques, **53**, January 1929.
8 Jahresbericht der deutschen Mathematiker Vereinigung, **37**, 11, 12.
9 Japanese Journal of Mathematics, **5**, 4. **6**, 1.
10 Journal fur die reine und angewandte Mathematik (Crelle), **160**, 2.
11 Mathematical Gazette, **14**, 199 (3 copies).
12 Mathematische Annalen, **101**, 1, 2, 3.
13 Monthly Notices of the Royal Astronomical Society, **89**, 4.
14 Niew Archief voor Wiskunde, **16**, of 1929.
15 Philosophical Magazine. **7**, 44, 45.
16 Popular Astronomy, **37**. 3, 4.
17 Proceeding of the Cambridge Philosophical Society, **25**, 1.
18 Proceedings of the London Mathematical Society, **29**, 4.
19 Universidad Nacional de La Plata, No. 86.
20 Proceedings of the Physico-Mathematical Society of Japan.
21 Proceedings of the Royal Society of London, **123**, 791, 792. (2 copies).
22 Rendiconti del circolo Mathematico di Palermo, **53**, 1.
23 Tohoku Mathematical Journal, **30**, 3, 4.
24 Transactions of the Royal Society of South Africa, **17**, Part 2.
25 Engineering Herald, **1**, 8.

Books.

Bombay University Calendar 1927-28 and 1928-29.

Report on the Statistical Analysis of the Medical Examination 1924 to 1925 and 1926 to 1927 of students of the Mysore University.

Reprints from the *Journal of the London Mathematical Society*:

Theorems stated by Ramanujan.

1 Theorems on Integrals, by C. T. Preece.
2 On Summation of Series, by G. N. Watson.
3 Theorems on Transformations of Series and Integrals, by C. T. Preece.
4 Theorems on Approximate Integration and Summation of Series, by G. N. Watson.

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Numbers in black type refer to the volumes, and those in ordinary type to the numbers of the issues.

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